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NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF DIFFERENCE EQUATIONS

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ABSTRACT. Consider the difference equation

$$A_{n+1} - A_n + \sum_{j=1}^{m} q_j A_{n-k_j} = 0, \quad n = 0, 1, 2, \dots \tag{1}$$

where for $j = 1, 2, \dots, m$ the coefficients q_j are positive real numbers and the indices k_j are nonnegative integers. We give an elementary proof of the (not so well-known) result that "every solution of Eq. (1) oscillates if and only if the characteristic equation

$$\lambda - 1 + \sum_{j=1}^{m} q_j \lambda^{-k_j} = 0$$

has no positive roots."

1. INTRODUCTION.

Chaos and fractals are at the center of attention nowadays and difference equations is what gives birth to both of them. Our aim in this note is to establish, by an extremely simple method, a necessary and sufficient condition for the oscillation of all solutions of the difference equation

$$A_{n+1} - A_n + \sum_{j=1}^{m} q_j A_{n-k_j} = 0, \quad n = 0, 1, 2, \dots \tag{1}$$

where

$$q_j \in (0, \infty) \text{ and } k_n \in \mathrm{N} = \{0, 1, 2, \dots\} \text{ for } j = 1, 2, \dots, m \tag{2}$$

in terms of the characteristic equation

$$\lambda - 1 + \sum_{j=1}^{m} q_j \lambda^{-k_j} = 0. \qquad (3)$$

More precisely, we will prove the following result.

Theorem. *Assume that* (2) *holds. Then the following statements are equivalent:*

(a) *Every solution of Eq.* (1) *oscillates.*

(b) *The characteristic equation* (3) *has no positive roots.*

By a *solution* of Eq. (1) we mean a sequence $\{A_n\}$ which is defined for $n \geq -k$, where $k = \max_{1\leq j\leq m} k_j$ and which satisfies Eq. (1) for $n \geq 0$.

A solution $\{A_n\}$ of Eq. (1) is called *nonoscillatory* if the terms A_n of the sequence are eventually positive or eventually negative. Otherwise the solution is called *oscillatory*.

A proof of Theorem 1, even in the more general case where the q_j's are real numbers and the k_j's are integers, can be given by a detailed analysis of the representation of every solution of Eq. (1) in terms of the characteristic roots of Eq. (3). See [1] and [3]. But this analysis is lengthy and sophisticated and makes use of a substantial amount of knowledge about the solutions of linear difference equations. Our proof, on the other hand, requires absolutely no prior familiarity with the theory of difference equations. The deepest result that we will use is that "a positive and continuous function on a closed and bounded set has a positive minimum."

2. THE PROOF OF THEOREM 1.

(a) $\Rightarrow$ (b). Otherwise, Eq. (2) has a positive root λ. But then $A_n = \lambda^n$ is a nonoscillatory solution of Eq. (1) which is a contradiction.

(b) $\Rightarrow$ (a). Assume, for the sake of contradiction, that Eq. (1) has a nonoscillatory solution $\{A_n\}$. As the opposite of a solution of Eq. (1) is also a solution, we may (and do) assume that $\{A_n\}$ is eventually positive. Then eventually,

$$A_{n+1} - A_n = -\sum_{j=1}^{m} q_j A_{n-k_j} < 0$$

and so $\{A_n\}$ is eventually decreasing. If $k_j = 0$ for all $j = 1, 2, \ldots, m$, then from (1) it follows that

$$A_{n+1} - \left(1 - \sum_{j=1}^{m} q_j\right) A_n = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

and so $\lambda_0 = 1 - \sum_{j=1}^{m} q_j > 0$. But in this case Eq. (3) has the positive root λ_0, which is impossible. Hence, we will suppose that for at least one $j \in \{1, 2, \ldots, m\}$, $k_j > 0$.

Define the set

$$\Lambda = \{\lambda \in (-\infty, \infty) : A_{n+1} - \lambda A_n \leq 0\}$$

with the convention that all the inequalities in this paper, which involve n, are assumed to be true eventually for all sufficiently large n. Clearly,

$$1 \in \Lambda \quad \text{and} \quad (-\infty, 0] \cap \Lambda = \emptyset.$$

Therefore, the proof will be complete if we establish the following claim: There exists a positive number r such that

$$\lambda \in \Lambda \Rightarrow \lambda - r \in \Lambda. \tag{4}$$

To this end, let us first define the number r. Set

$$F(\lambda) = \lambda - 1 + \sum_{j=1}^{m} q_j \lambda^{-k_j} \quad \text{for} \quad \lambda \in \mathrm{R} - \{0\}$$

and observe that $F(\infty) = \infty$ and $F(0+) = \infty$. As $F(\lambda) = 0$ has no positive roots, it follows that

$$r \equiv \min_{\lambda > 0} F(\lambda)$$

exists and is positive. Clearly,

$$-1+\sum_{j=1}^{m} q_j\lambda^{-k_j} \geq r-\lambda \quad \text{for } \lambda > 0. \tag{5}$$

Now, let $\lambda \in \Lambda$. Then for every $j = 1, 2, \ldots, m$ and for n sufficiently large,

$$A_{n+1} \leq \lambda A_n \quad \text{or} \quad A_{n-1} \geq \lambda^{-1} A_n$$

and by induction

$$A_{n-k_j} \geq \lambda^{-k_j} A_n. \tag{6}$$

Hence, from (1), (6) and (5) we see that

$$\begin{aligned}
0 &= A_{n+1} - A_n + \sum_{j=1}^{m} q_j A_{n-k_j} \\
&\geq A_{n+1} - A_n + \sum_{j=1}^{m} q_j \lambda^{-k_j} A_n \\
&= A_{n+1} + \left(-1 + \sum_{j=1}^{m} q_j \lambda^{-k_j}\right) A_n \\
&\geq A_{n+1} + (r-\lambda) A_n \\
&= A_{n+1} - (\lambda - r) A_n
\end{aligned}$$

which proves (4). The proof of the theorem is complete.

Remark. One can see that for $j = 1, 2, \ldots, m$

$$\inf_{0<\lambda<1} \left[\frac{1}{(1-\lambda)\lambda^{k_j}}\right] = \frac{(k_j+1)^{k_j+1}}{k_j^{k_j}}.$$

(Here we use the convention that $0^0 = 1$.) Then for $0 < \lambda < 1$,

$$\begin{aligned}
\lambda - 1 + \sum_{j=1}^{m} q_j \lambda^{-k_j} &= (1-\lambda)\left[-1 + \sum_{j=1}^{m} q_j \frac{1}{(1-\lambda)\lambda^{k_j}}\right] \\
&\geq (1-\lambda)\left[-1 + \sum_{j=1}^{m} q_j \frac{(k_j+1)^{k_j+1}}{k_j^{k_j}}\right].
\end{aligned}$$

As Eq. (3) has no roots in $[1, \infty)$, it follows that if (2) holds and

$$\sum_{j=1}^{m} q_j \frac{(k_j+1)^{k_j+1}}{k_j^{k_j}} > 1 \tag{7}$$

then every solution of Eq. (1) oscillates.

Condition (7) should be viewed as the discrete analogue of the condition of Hunt and Yorke [2],

$$\sum_{j=1}^{m} q_j \tau_j e > 1$$

for the oscillation of all solutions of the delay differential equation

$$\dot{x}(t) + \sum_{j=1}^{m} q_j x(t-\tau_j) = 0$$

where

$$q_j \in (0,\infty) \quad \text{and} \quad \tau_j \in [0,\infty) \quad \text{for } j = 1,2,\ldots,m.$$

The analogy can be easily seen if we observe that

$$\frac{(k+1)^k}{k^k} = \left(1+\frac{1}{k}\right)^k \to e, \quad \text{as } k \to \infty.$$

REFERENCES

1. I. Györi and G. Ladas, Linearized oscillations of differential equations with piecewise constant arguments, *Differential and Integral Equations* (to appear).

2. B. R. Hunt and J. A. Yorke, When all solutions of $x' = -\sum_{i=1}^{n} q_i(t)x(t - T_i(t))$ oscillate, *J. Differential Equations* 53 (1984), 139–145.

3. E. C. Partheniadis, Stability and oscillation of neutral delay differential equations with piecewise constant argument, *Differential and Integral Equations* (to appear).