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# INTEGRAL REPRESENTATION OF SOLUTIONS OF LINEAR ABSTRACT VOLTERRA FUNCTIONAL DIFFERENTIAL EQUATIONS

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## 1. Introduction.

The aim of this paper is to provide integral representation formulas (variation of parameters type) for the solutions of the linear abstract Volterra functional-differential equations of the form

$$\dot{x}(t)=(Lx)(t)+f(t), \tag{1}$$

in which $L : L^1_{loc}(R_+, R^n) \to L^1_{loc}(R_+, R^n)$ is a linear continuous operator of Volterra type (causal, nonanticipative).

This problem has been discussed under the main assumption that $L : L^2_{loc}(R_+,R^n) \to L^2_{loc}(R_+,R^n)$ is linear, continuous, and causal, in our book [2] and in the paper [3]. In the present paper, we consider a different setting for the problem of representation of solutions of linear abstract Volterra functional-differential equations, choosing as underlying space the space $L^1_{loc}(R_+,R^n)$, instead of $L^2_{loc}(R_+,R^n)$. This apparently minor change does require certain modifications of the procedure used in [2], [3], and leads to results which differ from those obtained in the papers mentioned above.

The procedure we will use consists in reducing first the equation (1), together with an initial condition of the form

$$x(0)=x^0 \in R^n, \tag{2}$$

to an integral equation of classical Volterra type. The existence of a resolvent kernel for the later enables us to obtain the representation formula for solutions in the form

$$x(t)=X(t, 0)x^0+\int_0^t X(t, s)\, f(s)ds, \tag{3}$$

where the matrix $X(t, s)$, the so-called transition operator, is uniquely determined by the operator L in (1).

Formula (3) is then applied to obtain variation of parameters formulas for certain classes of delay-differential equations, in which functional initial data are involved. As emphasized in [2], [3], this procedure appears as a unifying tool in treating various classes of delay equations.

We will also indicate a few other applications of the formula (3) and its consequences to the solution of boundary value problems attached to functional-differential equations of Volterra type, to control theory, and to different other kind of problems (stability, asymptotic behavior).

## 2. Existence and Uniqueness.

We shall prove first that equation (1), with the initial condition (2), has a unique solution $x=x(t)$, locally absolutely continuous on $R_+$. The following hypotheses are general enough, to encompass a large number of significant special cases:

a) $L : L^1_{loc}(R_+, R^n) \to L^1_{loc}(R_+, R^n)$ is linear, continuous, and of Volterra type.

b) $f \in L^1_{loc}(R_+, R^n)$.

Theorem 1. Under assumptions a) and b) stated above, there exists a unique solution $x=x(t)$ of the equation (1), i.e., a locally absolutely continuous map $x : R_+ \to R^n$, verifying (1) a.e., such that (2) holds.

Proof. As usual, we integrate both sides of (1) from 0 to t, $t>0$, and take into account (2). One obtains

$$x(t)=x^0+\int_0^t f(s)ds+\int_0^t (Lx)(s)ds, \tag{4}$$

which is, under our assumptions, equivalent to (1), (2). If we can prove for (4) the existence and uniqueness of a locally absolutely continuous solution, then Theorem 1 is proven. It is useful to notice that the right hand side of (4) is locally absolutely continuous, for any $f$, $x \in L^1_{loc}(R_+, R^n)$.

Therefore, we can concentrate on (4) in order to prove Theorem 1. Moreover, if we can prove that (4) has a unique continuous solution on $R_+$, it follows that such a solution will be locally absolutely continuous on $R_+$, and verifies (1), (2).

We can choose as underlying space for the proof, the space $C(R_+, R^n)$ of all continuous maps from $R_+$ into $R^n$, with the usual topology of uniform convergence on any compact interval of $R_+$. In this space, we easily obtain the convergence of the successive approximations defined by

$$x^m(t) = x^0 + \int_0^t f(s)ds + \int_0^t (Lx^{m-1})(s)ds, \tag{5}$$

for any $m \geq 1$. On behalf of our assumptions all $x^m(t)$, $m=0, 1, 2, \cdots$, are defined on $R_+$, taking values in $R^n$. Moreover, each $x^m(t)$, $m=0, 1, 2, \cdots$, is locally absolutely continuous on $R_+$, and consequently belongs to $C(R_+, R^n)$.

The recurrent formula (5) and the linearity of L lead to

$$x^{m+1}(t) - x^m(t) = \int_0^t (L(x^m - x^{m-1}))(s)ds, \tag{6}$$

valid for $t \in R_+$, and $m \geq 1$. Since L is continuous from $L^1_{loc}(R_+, R^n)$ into itself, the equation (6) implies the inequality

$$\left|x^{m+1}(t) - x^m(t)\right| \leq M \int_0^t \left|x^m(s) - x^{m-1}(s)\right| ds \tag{7}$$

on any interval $[0, T]$, $T < +\infty$, with $M = M(T) > 0$, and $m \geq 1$. Indeed, we know that $x \to Lx$ is continuous from $L^1_{loc}(R_+, R^n)$ into itself, while $y \to \int_0^t y(s)ds$ is continuous from $L^1_{loc}(R_+, R^n)$ into $C(R_+, R^n)$. It is a simple matter to see that, due to the continuity of the operator L, the function

M(T) is nondecreasing in T.

If we notice that on a fixed interval [0, T] one has

$$|x^1(t)-x^0|\leq A, \tag{8}$$

with $A=A(T)>0$ sufficiently large, then (7) leads to the usual estimate

$$\left|x^{m+1}(t)-x^m(t)\right|\leq A\,\frac{(Mt)^m}{m!}\,,\ t\in[0,\,T], \tag{9}$$

valid for $m\geq 0$. Therefore, the sequence $\{x^m(t)\}$ converges uniformly on any finite interval [0, T], i.e., it converges in $C(R_+,\,R^n)$. If we define

$$x(t)=\lim_{n\to\infty}x^m(t), \tag{10}$$

there results from (5) that $x(t)$ is a solution of (4), defined an $R_+$ and continues there. As noticed above, $x(t)$ is actually locally absolutely continuous on $R_+$, which ends the proof of the existence part of Theorem 1. The uniqueness is obtained using also the successive approximations defined by (5), in the standard manner. Of course, it can be also obtained by using the Gronwall's type inequality on any [0, T]. The Theorem 1 is thereby proven.

Remark 1. Instead of assuming that $L:L^1_{loc}(R_+,\,R^n)\to L^1_{loc}(R_+,\,R^n)$, it is possible to replace $L^1_{loc}(R_+,\,R^n)$ by $L^p_{loc}(R_+,\,R^n)$ as range, with $1\leq p<\infty$. The proof remains, basically, the same.

Remark 2. We shall see in the next section that equation (4) can be written in the form

$$x(t)=g(t)+\int_0^t k(t,\,s)x(s)ds, \tag{11}$$

where

$$g(t)=x^0+\int_0^t f(s)ds, \tag{12}$$

and $k(t,s)$ is a measurable kernel whose existence has to be proven, such that

$$\int_0^t (Lx)(s)ds=\int_0^t k(t,s)x(s)ds. \tag{13}$$

Then, equation (11) can be dealt with in the classical manner, and global existence (on $R_+$) can be obtained either by successive approximations or by the fixed point method. For details in regard to this alternate approach see [2], [3].

## 3. The Integral Representation of Solutions

The key fact in obtaining an integral representation formula for the solutions of (1), (2), or equivalently for equation (4) is the validity of a representation formula of the form (13), in which L stands for a linear abstract Volterra operator, acting from $L^1_{loc}(R_+, R^n)$ into itself.

A formal representation of the form (13) can be obtained relatively easy if we rely on some basic results concerning integral operators. More specifically, if we notice that the map

$$x \to \int_0^t (Lx)(s)ds \tag{14}$$

from $L^1_{loc}(R_+, R^n)$ into $C(R_+, R^n)$ is continuous, and if we fix now a $t>0$, then (14) defines a linear continuous functional from $L^1([0, t], R^n)$ into $R^n$. We know (see for instance [5]) that such a map can be represented in the form (13), for each $t>0$, where $k(t, s)$ is in $L^\infty$ with respect to $s : k(t,\cdot) \in L^\infty([0, t], \mathcal{L}(R^n, R^n))$.

The kernel $k(t, s)$, as obtained above by direct application of the well known fact that the dual space of $L^1$ is $L^\infty$, may not be measurable as a function of $(t, s)$, $0 \le s \le t < \infty$. At this point, a result of Bukhvalov, reproduced in [5], can be used, in order to conclude the fact that in the representation (13) we can always substitute $k(t, s)$ by a measurable kernel.

Hence, for any linear abstract Volterra operator L, continuous on $L^1_{loc}(R_+, R^n)$, we can infer the existence of a measurable kernel $k(t, s)$, such that (13) takes place. Moreover, for every fixed $t>0$, $k(t, \cdot)$ is essentially bounded on $[0, t]$. This property alone of $k(t, s)$ does not allow us to construct the resolvent kernel $\tilde{k}(t,s)$, associated with $k(t, s)$. In order to obtain more information about the kernel $k(t,s)$, information which is necessary in investigating the linear Volterra integral equation (11), we will rely on the continuity of the map (14), from $L^1_{loc}(R_+, R^n)$ into $C(R_+, R^n)$. This property has been noticed in the proof of Theorem 1.

The continuity of the map (14)

$$x \to \int_0^t (Lx)(s)ds = \int_0^t k(t,s)x(s)ds,$$

from $L^1_{loc}(R_+, R^n)$ into $C(R_+, R^n)$ implies certain properties for the kernel $k(t,s)$, as pointed out for the first time by Radon (see [6]). Namely, for every $t>0$, one must have

$$\operatorname*{ess\text{-}sup}_{0\leq s\leq t}\|k(t, s)\| = K_t < \infty, \tag{15}$$

besides other conditions which are not significant for the development of our problem.

What is really important from our point of view is the fact that $K_t$, occurring in (15), is also essentially bounded on any interval $[0, T]\subset R_+$. In other words, the kernel $k(t, s)$ in (13) is essentially bounded on any set (triangle) $0\leq s\leq t\leq T<\infty$, or, it is locally bounded in the set

$$\Delta = \{(t, s);\ 0\leq s\leq t<\infty\}.$$

Of course, this means $k\in L^\infty_{loc}(\Delta, \mathcal{L}(R^n, R^n))$.

It turns out that any kernel belonging to $L^\infty_{loc}(\Delta, \mathcal{L}(R^n, R^n))$ has a resolvent kernel which can be constructed according to classical procedure: if $k\in L^\infty_{loc}(\Delta, \mathcal{L}(R^n, R^n))$ is given, then one denotes $k_1(t, s) = k(t, s)$, and define

$$\tilde{k}(t, s) = \sum_{j=1}^{\infty} k_j(t,s), \quad (t,s)\in\Delta, \tag{16}$$

where

$$k_{j+1}(t,s) = \int_s^t k(t,u)k_j(u,s)du, \ j\geq 1. \tag{17}$$

The series in (16) does converge in $L^{\infty}_{loc}(\Delta, \mathcal{L}(R^n, R^n))$, as it can be easily checked which implies that $\tilde{k} \in L^{\infty}_{loc}(\Delta, \mathcal{L}(R^n, R^n))$.

If we return now to the equation (11), then its unique (continuous) solution can be represented in the form

$$x(t)=g(t)+\int_0^t \tilde{k}(t, s)g(s)ds. \tag{18}$$

But g(t) is given by the formula (12), and if we substitute now in (18) we obtain formula (3) for x(t), in which

$$X(t, s)=I+\int_0^t \tilde{k}(t, u)du, \ (t, s)\in\Delta. \tag{19}$$

The representation formula (3) is a source of more properties for the matrix kernel X(t,s) defined by (19). This aspect will be considered later.

In concluding this section, the following result can be stated in regard to our initial value problem (1), (2), which has been shown to be equivalent to (11):

Theorem 2. Under assumptions a) and b) formulated in Section 2, the unique solution $x \in AC_{loc}(R_+, R^n)$ of the problem (1), (2) can be expressed by means of the integral formula (3), where X(t,s) is defined in terms of the resolvent kernel $\tilde{k}(t, s)$ of (11) by (19).

The proof of Theorem 2 has been provided above.

Remark. If instead of operators acting from $L^1_{loc}(R_+, R^n)$ into itself, we consider operators from $L^1_{loc}(R_+, R^n)$ into $L^p_{loc}(R_+, R^n)$, $1<p<\infty$, then the map (14) is compact. This feature implies more properties for the kernel k(t, s) than used in the proof of Theorem 2. Such properties are useful in further investigation of the representation (3).

## 4. Application to Delay-Differential Equations.

The result in Theorem 2 leads easily to a variation of constant formula for delay-differential equations of the form

$$\dot{x}(t)=L(t,x_t)+f(t),$$

under initial conditions

(21) $$x_0=\varphi\in S,\ x(0)=x^0\in R^n.$$

As usual, $x_t(s)=x(t+s)$, $s\in(-h, 0]$, with $h>0$ or $h=\infty$ (infinite delay), the space $S$ consisting of (at least measurable) maps from $(-h, 0]$ into $R^n$. For every $t\in R_+$, $L(t, \varphi)$ stands for an operator defined on $S$, with values in $R^n$, while the map

(22) $$t\to L(t, x_t)$$

from $R_+$ into $L^1_{loc}(R_+, R^n)$ satisfies certain conditions, to be specified. Let us notice that we must have $x_t\in S=$ the space of initial functions, for any $x\in L^1_{loc}(R_+, R^n)$ and $t\in R_+$. Of course, this condition implies the local integrability of the functions in S.

The manner in which $x_t$ has been defined above for the $t\in R_+$ and $x\in L^1_{loc}(R_+, R^n)$ shows that the map

(23) $$x\to L(t, x_t),\ t\in R_+,$$

from $L^1_{loc}(R_+, R^n)$ into itself is linear, when $L(t, \varphi)$ is linear in the second argument. and is of Volterra type. In other words, if two elements $x, y\in L^1_{loc}(R_+, R^n)$ are such that $x(t)\equiv y(t)$ a.e. on $[0, T]$, then $L(t, x_t)\equiv L(t, y_t)$ a.e. on the same interval. Of course, this occurs because on $(-h, 0]$, each $x$ is extended by means of the function $\varphi\in S$, appearing in the initial conditions (21).

Therefore, it remains to formulate adequate conditions on $L(t, \varphi)$ and the space $S$ of initial data, such that hypothesis a) of Theorems 1 and 2 be satisfied. Then both Theorems 1 and 2 can be applied, and a variation of parameters formula results for the initial value problems (20), (21).

A convenient set of conditions in order to place ourselves under the hypotheses of Theorems 1 and 2 consists of the following:

1. For every $t\in R_+$, $L(t, \cdot)$ is a linear map from $S$ into $R^n$, such that

(24) $$|L(t, \varphi)|\le\lambda(t)|\varphi|_S,\ t\in R_+,$$

Where $\lambda$ is a nonnegative locally $L^{\infty}$ – function on $R_+$.

2. The "phase space" $S = S((-h, 0], R^n)$ is a Banach space or a Fréchet space of locally integrable functions, in which the norm or the invariant metric is denoted by $|\cdot|_S$, the following conditions being satisfied:

α) for every $\varphi \in S$ and $x \in L^1_{loc}(R_+, R^n)$, the map $t \to x_t$ from $R_+$ into $S$ is continuous;

β) there exists a nonnegative locally $L^{\infty}$-function $m(t)$, such that

(25) $$\left|\varphi(t+u)\chi_{(-h,\,-t)}(u)\right|_S \le m(t)|\varphi|_S,$$

for almost all $t \in R_+$;

γ) there exists a nonnegative locally $L^{\infty}$-function $n(t)$, such that

(26) $$\left|x_t(u)\chi_{(-t,\,0)}(u)\right|_S \le n(t)\int_0^t |x(u)|du,$$

a.e. on $R_+$, for all $x \in L^1_{loc}(R_+, R^n)$.

A simple example of a space $S$ satisfying the conditions listed above is $S = L^1((-h, 0], R^n)$. For more details in this respect, see [2].

If we noticed that the definition of $x_t$ implies the representation

(27) $$x_t(u) = x_t(u)\chi_{(-t,\,0]}(u) + \varphi(t+u)\chi_{(-h,\,-t]}(u)$$

for $u \in (-h, 0]$ and $t \in R_+$, then the equation (20) becomes

(28) $$\dot{x}(t) = L(t, x_t\chi_{(-t,\,0]}) + L(t, \varphi(t+\cdot)\chi_{(-h,\,-t]}(\cdot)) + f(t).$$

Applying formula (3) to this equation, in which the first term into the right hand side is the linear volterra operator, one obtains

(29) $$x(t) = X(t, 0)x^0 + \int_0^t X(t, s)f(s)ds + \int_0^t X(t,s)L(s, \varphi(s+\cdot)\chi_{(-h,-s)}(\cdot))ds,$$

which constitutes the desired variation of parameters formula for the problem (20), (21).

As a typical example of an operator $L(t, \varphi)$, one can consider the integral operator

$$L(t, \varphi)=\int_{-h}^{0} A(t, s)\varphi(s)ds, \tag{30}$$

in which $A(t, s)$ is a measurable matrix-valued kernel $(n\times n)$, such that, for fixed $t$, $\|A(t, \cdot)\|$ is in $L^{\infty}((-h,0],R)$, and

$$\int_{-h}^{0} \|A(t, s)\|ds \in L^{1}_{loc}(R_{+}, R).$$

In order to cover such operators as

$$L(t, \varphi)=A(t)\varphi(0)+B(t)\varphi(-\tau),\ 0<\tau<h,$$

where $A$ and $B$ are matrices, one should integrate in (30) with respect to a family of measures.

## 4. Various Applications.

As mentioned in the introductory section, the formula (3) providing the integral representation for the solutions of the abstract Volterra functional-differential equation (1), under initial condition (2) can be used to reach different purposes.

A first application we want to indicate relates to the behavior of solutions of the perturbed equation

$$\dot{x}(t)=(Lx)(t)+(fx)(t),\ t\in R_{+}, \tag{31}$$

where $L$ is a linear Volterra operator as in preceding sections, while $f$ stands for another operator acting on convenient function spaces, in general, nonlinear.

If we attach to (31) the usual initial condition (2), then by means of the representation formula (3) we obtain the following nonlinear functional equation for the unknown $x$:

$$x(t)=X(t,0)x^{0}+\int_{0}^{t} X(t,s)(fx)(s)ds. \tag{32}$$

The investigation of equation (32) is facilitated by the knowledge of properties of the transition matrix $X(t, s)$. Of course, as long as nothing is assumed on the linear system (1), very little can be said about $X(t, s)$-especially, the asymptotic behavior as $t\to\infty$.

Let us assume, for instance, that (1) is such that its solution corresponding to $x^0=\theta=$the zero vector in $R^n$, is bounded on $R_+$, for any $f\in L^\infty(R_+, R^n)$. The reader familiar with stability theory, will recognize that this assumptions is, in fact, a stability condition on the homogeneous system $\dot{x}(t)=(Lx)(t)$.

Since the unique solution of (1), with zero initial condition, is given by

$$x(t)=\int_0^t X(t, s)f(s)ds, \tag{33}$$

and this solution must belong to $L^\infty(R_+, R^n)$ for any f in the same space, one derives (see [1], Sec. 26) the condition

$$\int_0^t \|x(t, s)\|ds\leq M<+\infty, \quad \text{a.e. on } R_+, \tag{34}$$

with $M$ a positive number. Condition (34) is well-known in the System-theoretic literature as the BIBO (bounded imput, bounded output) condition of stability.

If we now take as underlying space for the investigation of equation (32), the space $BC(R_+, R^n)$ of all continuous bounded maps from $R_+$ into $R^n$, with the usual supremum norm, then it is easily seen that the operator

$$(Tx)(t)=X(t,0)x^0+\int_0^t X(t,s)(fx)(s)ds \tag{35}$$

is acting from $BC(R_+, R^n)$ into itself, provided f is acting on that space. If we also assume

$$|fx-fy|_{BC}\leq\lambda\,|x-y|_{BC}, \tag{36}$$

then one obtains

$$|Tx-Ty|_{BC}\leq M\lambda\,|x-y|_{BC}, \quad x, y\in BC, \tag{36}$$

which implies that T is a contraction as soon as $M\lambda<1$.

Of course, this result was to be expected. However, it provides a simple illustration of the usefulness of formula (3).

Another application we want to consider is related to boundary value problems associated with equation (1). This time, we will assume that the operator L is acting from $L^1([0, T], R^n)$ into itself, with $T>0$ a fixed number.

Instead of the initial condition (2), we can impose to the solution a linear restriction of the form

$$Ax(0)+Bx(T)=c, \tag{38}$$

where A, B are square matrices of order n, and c is in $R^n$.

Using again formula (3) for the solution, we want to determine $x^0$ (the initial value of the solution), such that (38) holds true. This implies

$$[A+BX(T,0)]x^0=c-B\int_0^T X(T,s)f(s)ds, \tag{39}$$

which means that under the assumption

$$\det[A+BX(T, 0)]\neq 0, \tag{40}$$

the linear system (1), with the boundary value condition (38), has a unique solution. In other words, $x^0$ can be uniquely determined from (39).

If we substitute $x^0$ from (39) into formula (3), then the following expression is obtained for the solution of the problem (1), (38):

$$x(t)=x_0(t)+\int_0^T \tilde{X}(t, s)f(s)ds, \tag{41}$$

where

$$x_0(t)=X(t, 0)\,[A+BX(T, 0)]^{-1}c, \tag{42}$$

and

$$\tilde{X}(t, s)=\begin{cases} X(t,s)-X(t, 0)[A+BX(T,0)]^{-1}BX(t, s), & 0\leq s\leq t, \\ -X(t,0)[A+BX(T,0)]^{-1}BX(T, s), & t\leq s\leq T. \end{cases} \tag{43}$$

If we consider the nonlinear equation (31), under condition (38), then the following Fredholm type equation is obtained from (41),

$$x(t)=x_0(t)+\int_0^T \tilde{X}(t, s)(fx)(s)ds. \tag{44}$$

Equation (44) can be dealt with in the same manner we proceeded with equation (32). This time, we can take the space $C([0, T], R^n)$ as underlying space. Actually it is possible to deal only with operators $f$ acting on this space, even though $L$ is acting on a much larger space. Contraction mapping theorem, or results on the existence of fixed point under weak compactness assumptions can be applied to obtain existence of solutions to (44).

A third type of application, for the representation formula (3), is related to control problems. Namely, if we consider the system

$$\dot{x}(t)=(Lx)(t)+B(t)u(t),\ t\in R_+, \tag{45}$$

under the initial condition (2), then under adequate assumptions we can represent the "output" $x(t)$ in terms of the "input" $u(t)$ by the formula derived from (3):

$$x(t)=X(t, 0)x^0+\int_0^t X(t, s)B(s)u(s)ds. \tag{46}$$

Since $X(t, s)$ is in $L^{\infty}_{loc}(\Delta, \mathcal{L}(R^u, R^n))$, fairly mild assumptions can be made on $B$ and the controller $u$, in order to obtain results regarding the system (45).

Let us consider, as an illustration, the BIBO stability problem. For simplicity, we will assume the initial condition to be $x(0)=\theta$, which reduces the formula (46) to the form

$$x(t)=\int_0^t X(t, s)B(s)u(s)ds. \tag{46}$$

Since the controller $u$ is usually assumed to be bounded, i.e., in the space $L^{\infty}(R_+, R^m)$, the BIBO stability condition is very much alike (34):

$$\int_0^t \|X(t, s)B(s)\|ds\le M<\infty,\ \text{a.e. on } R_+. \tag{48}$$

The matrix $B(t)$, of type $n\times m$, must be measurable. If $\|X(t, s)\|$ is in $L^{\infty}(\Delta, R)$, then condition

(48) will be satisfied under the only assumption $B(t) \in L^1(R_+, \mathcal{L}(R^n, R^n))$.

For different other applications, in particular to Optimal Control problems, see [4].

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