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Some Linear and Nonlinear Integral Inequalities

Adrian CORDUNEANU

Abstract. The method of the resolvent kernel and the Bellman-Bihari procedure are used to obtain bounds for the functions satisfying some inequalities involving iterated integrals. The results presented below may be in a natural way extended to the case of several independent variables. An application is given to a Darboux-type problem for a partial differential equation.

We consider the integral inequality

$$u(x) \leq K + \int_x^\alpha a(s)\, g(u(s))\, ds + \int_x^\alpha \left(\int_s^\alpha b(s,t)\, g(u(t))\, dt \right) ds + \\ + \int_x^\alpha \left(\int_s^\alpha \left(\int_t^\alpha c(s,t,\tau)\, g(u(\tau))\, d\tau \right) dt \right) ds, \quad x \in [\alpha, \alpha] \tag{1}$$

where the function $u = u(x)$ and $a = a(x)$ are defined on $[\alpha_1, \alpha]$, $b = b(x,s)$ is defined for $\alpha_1 \leq x \leq s \leq \alpha$, $c = c(x,s,t)$ is defined for $\alpha_1 \leq x \leq s \leq t \leq \alpha$. In what foolows, we shall use the notation

$$H(x) = \int_x^\alpha a(s)\, ds + \int_x^\alpha \int_s^\alpha b(s,t)\, ds\, dt + \int_x^\alpha \int_s^\alpha \int_t^\alpha c(s,t,\tau)\, ds\, dt\, d\tau \tag{2}$$

for $x \in [\alpha_1, \alpha]$. We can state

Proposition 1. *Assume that all the function involved in* (1) *are continuous, nonnegative and that $K > 0$ is a constant. Also assume that $g = g(u)$ is nondecreasing, defined for $u \geq 0$ and $g(u) > 0$ for $u > 0$. Then we have*

$$u(x) \leq G^{-1}(H(x)) \quad \text{provided that} \quad H(x) < G(\infty), \tag{3}$$

where $G = G(u)$ is the primitive of $1/g$, satisfying the condition $G(K) = 0$.

PROOF. Denoting by $v = v(x)$ the right-hand side of (1), it folows that v is nonincreasing, $u(x) \leq v(x)$ on $[\alpha_1, \alpha]$ and $v(\alpha) = K$. To obtain an evaluation for $v = v(x)$, we shall make use of the derivative

$$v'(x) = -a(x)\, g(u(x)) - \int_x^\alpha b(x,t)\, g(u(t))\, dt - \int_x^\alpha \int_t^\alpha c(x,t,\tau)\, g(u(\tau))\, dt\, d\tau \tag{4}$$

for $x \in [\alpha_1, \alpha]$. We remark that $a(x)\, g(u(x)) \leq a(x)\, g(v(x))$ on the interval $[\alpha_1, \alpha]$. We also have $b(x,t)\, g(u(t)) \leq b(x,t)\, g(v(x))$, using the assumptions that g is nondecreasing and v is nonincreasing. For $\alpha_1 \leq x \leq t \leq \tau \leq \alpha$, we have $c(x,t,\tau)\, g(u(\tau)) \leq c(x,t,\tau)\, g(v(\tau)) \leq$ $\leq c(x,t,\tau)\, g(v(x))$, using again the monotonicity of the functions g and v. Now, from (4), we deduce the inequality

$$\frac{v'(x)}{g(v(x))} \geq -\left(a(x) + \int_x^\alpha b(x,t)\, dt + \int_x^\alpha \int_t^\alpha c(x,t,\tau)\, dt\, d\tau\right), \tag{5}$$

that holds for $x \in [\alpha_1, \alpha]$. Putting here $x = s$ and integrating the resulting inequality on the interval $[x, \alpha]$, it follow

$$G(v(\alpha)) - G(v(x)) \geq -H(x) \tag{6}$$

for every $x \in [\alpha_1, \alpha]$. Because $G(v(\alpha)) = G(K) = 0$, the preceding inequality may be written in the form $G(v(x)) \leq H(x)$ and thus the bound (3) holds true. We remark that, if $G(\infty) = \infty$, we have $u(x) \leq G^{-1}H(x)$ for all $x \in [\alpha_1, \alpha]$. We also remark that, if the inequality (1) is considered on the interval $(-\infty, \alpha]$, the obtained bound and the above proof remain true. ■

Corollary 1. (The linear case of (1)) *Assume that the hypotheses of Proposition 1 are fulfilled and that $g(u) = u$. Then it follows, that*

$$u(x) \leq K \exp(H(x)), \quad x \in [\alpha_1, \alpha] \tag{7}$$

where $H(x)$ is defined by (2)*.*

Remark 1. Using the employed here method of proof, we can show that from the inequality

$$u(x) \geq K + \int_x^\alpha a(s)\, g(u(s))\, ds, \quad x \in [\alpha_1, \alpha] \tag{8}$$

where u, a, g, G satisfy the hypotheses of Proposition 1 and $K > 0$ is a constant, it follows

$$u(x) \geq G^{-1}\left(\int_x^\alpha a(s)\, ds\right), \quad \text{provided that} \quad \int_x^\alpha a(s)\, ds < G(\infty). \tag{9}$$

In the case of two independent variables, a similar inequality to (1) is of the form

$$u(x,y) \leq K + \int_x^\alpha \int_y^\beta a(s,t)\, g(u(s,t)) + \int_x^\alpha \int_y^\beta \left(\int_s^\alpha \int_t^\beta b(s,t,\sigma,\tau)\, g(u(\sigma,\tau))\, d\sigma\, d\tau\right) ds\, dt \tag{10}$$

where the functions $u = u(x,y)$ and $a = a(x,y)$ are defined on $\Delta_1 = \{(x,y),\ \alpha_1 \leq x \leq \alpha,$ $\beta_1 \leq y \leq \beta\}$ and $b = b(x,y,s,t)$ is defined on $\Delta_2 = \{(x,y),\ \alpha_1 \leq x \leq s \leq \alpha, \beta_1 \leq y \leq t \leq \beta\}$. We can state the following

Proposition 2. *Assume that all the functions involved in* (10) *are continuous, nonnegative and that $K > 0$ is a constant. Also assume that g and G have the same properties as in*

Proposition 1. Then we have

$$u(x,y) \le G^{-1}(H(x,y)), \quad \textit{provided that} \quad H(x,y) < G(\infty) \tag{11}$$

where $H(x,y)$ is defined on Δ_1 and it is given by

$$H(x,y) = \int_x^\alpha \int_y^\beta a(s,t)\,ds\,dt + \int_x^\alpha \int_y^\beta \left(\int_s^\alpha \int_t^\beta b(s,t,\sigma,\tau)\,d\sigma\,d\tau \right) ds\,dt. \tag{12}$$

PROOF. Denoting by $v = v(x,y)$ the right-hand side of (10), it follows that v is nonincreasing with respect to each of its variables, $u(x,y) \le v(x,y)$ on Δ_1 and $v(x,\beta) = v(\alpha,y) = K$ for $x \in [\alpha_1,\alpha]$, $y \in [\beta_1,\beta]$. The partial derivative $\partial v/\partial x$ is given by

$$\frac{\partial v}{\partial x}(x,y) = -\int_y^\beta a(x,t)\,g(u(x,t))\,dt - \int_y^\beta \left(\int_x^\alpha \int_t^\beta b(x,t,\sigma,\tau)\,g(u(\sigma,\tau))\,d\sigma\,d\tau \right) dt \tag{13}$$

for $(x,y) \in \Delta_1$. In (13), for (x,y) fixed and $t \in [y,\beta]$, we have $a(x,t)\,g(u(x,t)) \le$ $\le a(x,t)\,g(v(x,t)) \le a(x,t)\,g(v(x,y))$, by using the monotonicity of the functions g and v. Regarding the last integral in (13), from $g(u(\sigma,\tau)) \le g(v(\sigma,\tau)) \le g(v(x,y))$ it follows $b(x,t,\sigma,\tau)\,g(u(\sigma,\tau)) \le b(x,t,\sigma,\tau)\,g(v(x,y))$. Then, from (13) we easily deduce the inequality

$$\frac{\dfrac{\partial v}{\partial x}(x,y)}{g(v(x,y))} \ge -\left(\int_y^\beta a(x,t)\,dt + \int_y^\beta \left(\int_x^\alpha \int_t^\beta b(x,t,\sigma,\tau)\,d\sigma\,d\tau \right) dt \right) \tag{14}$$

valid for $(x,y) \in \Delta_1$. Putting here $x = s$ and integrating the respective inequality on the interval $[x,\alpha]$ with respect to s ($y =$ fixed), we find

$$G(v(\alpha,y)) - G(v(x,y)) \ge -H(x,y) \tag{15}$$

holding for every $(x,y) \in \Delta_1$. Because $G(v(\alpha,y)) = G(K) = 0$, we obtain $G(v(x,y)) \le$ $\le H(x,y)$ which implies that (11) holds true. If $G(\infty) = \infty$, the restriction imposed in (11) is not necessary and we have $u(x,y) \le G^{-1}(H(x,y))$ for $(x,y) \in \Delta_1$. ■

Corollary 2. *Assume that, in* (10)*, the hypotheses of Proposition 2 are fulfilled and that $g(u) = u$. Then it follows that*

$$u(x,y) \le K \exp(H(x,y)), \quad (x,y) \in \Delta_1 \tag{16}$$

where $H(x,y)$ is given by (12)*.*

In what follows, we shall use the method of the resolvent kernel to obtain bounds in the case of linear integral inequalities, supposing that all involved functions are continuous and nonnegative. Let us consider the inequality

$$u(x,y) \le f(x,y) + \int_x^\alpha \int_y^\beta k(x,y,s,t)\,u(s,t)\,ds\,dt, \quad (x,y) \in \Delta_1 \tag{17}$$

where u, f are defined on $\Delta_1 = \{(x,y);\ \alpha_1 \le x \le \alpha, \beta_1 \le y \le \beta\}$ and k is defined on

$\Delta_2 = \{(x, y, s, t;\ \alpha_1 \le x \le s \le \alpha, \beta_1 \le y \le t \le \beta)\}$. It is well-known that we have $u(x, y) \le$ $\le v(x, y)$ on Δ_1, $v = v(x, y)$ being the solution of the corresponding linear integral equation (of Volterra type)

$$v(x, y) = f(x, y) + \int_x^\alpha \int_y^\beta k(x, y, s, t)\, v(s, t)\, ds\, dt, \quad (x, y) \in \Delta_1. \tag{18}$$

The solution $v = v(x, y)$ of this equation is unique and it may be obtained by the method of successive approximations, putting

$$v_0(x, y) = f(x, y),\ v_{n+1}(x, y) = f(x, y) + \int_x^\alpha \int_y^\beta k(x, y, s, t)\, v_n(s, t)\, ds\, dt \tag{19}$$

for $(x, y) \in \Delta_1$ and $n = 0, 1, 2, \ldots$. By using the notations

$$k_0(x, y, s, t) = k(x, y, s, t),\ k_{n+1}(x, y, s, t) = \int_x^s \int_y^t k(x, y, \sigma, \tau)\, k_n(\sigma, \tau, s, t)\, d\sigma\, d\tau \tag{20}$$

for $n = 0, 1, 2, \ldots$ and $(x, y, s, t) \in \Delta_2$, we obtain the formula

$$v_{n+1}(x, y) = f(x, y) + \int_x^\alpha \int_y^\beta \left(\sum_{i=0}^{n} k_i(x, t, s, t) \right) f(s, t)\, ds\, dt \tag{21}$$

for $n = 0, 1, 2, \ldots$ and $(x, y) \in \Delta_1$. Making $n \to \infty$, we obtain

$$v(x, y) = f(x, y) + \int_x^\alpha \int_y^\beta r(x, y, s, t)\, f(s, t)\, ds\, dt, \quad (x, y) \in \Delta_1 \tag{22}$$

where the resolvent kernel $r(x, y, s, t)$ is given by

$$r(x, y, s, t) = \sum_{i=0}^{n} k_i(x, y, s, t), \quad (x, y, s, t) \in \Delta_2. \tag{23}$$

In the case $k(x, y, s, t) = a(x, y)\, b(s, t)$, where a, b are continuous and nonnegative on Δ_1, we shall prove for the iterated k_n the bound

$$k_n(x, y, s, t) \le \frac{1}{n!} a(x, y)\, b(s, t) \left(\int_x^s \int_y^t a(\sigma, \tau)\, b(\sigma, \tau)\, d\sigma\, d\tau \right)^n \tag{24}$$

for $n \ge 1$ and $(x, y, s, t) \in \Delta_2$. To this end, we find

$$k_1(x, y, s, t) = a(x, y)\, b(s, t) \left(\int_x^s \int_y^t a(\sigma, \tau)\, b(\sigma, \tau)\, d\sigma\, d\tau \right) \tag{25}$$

on Δ_2 and we suppose that (24) is true for some $n \geq 1$. Then for k_{n+1}, we can write

$$k_{n+1}(x,y,s,t) \leq \frac{1}{n!} a(x,y)\, b(s,t) \int_x^s \int_y^t a(\sigma,\tau)\, b(\sigma,\tau) \left(\int_\sigma^s \int_\tau^t a(\xi,\eta)\, b(\xi,\eta)\, d\xi\, d\eta \right)^n d\sigma\, d\tau. \tag{26}$$

Now, for (x,y,s,t) fixed in Δ_2, we shall prove that

$$\begin{aligned} I = \int_x^s \int_y^t a(\sigma,\tau)\, b(\sigma,\tau) \left(\int_x^s \int_y^t a(\xi,\eta)\, b(\xi,\eta)\, d\xi\, d\eta \right)^n d\sigma\, d\tau \leq \\ \leq \frac{1}{n+1} \left(\int_x^s \int_y^t a(\sigma,\tau)\, b(\sigma,\tau)\, d\sigma\, d\tau \right)^{n+1}, \end{aligned} \tag{27}$$

denoting for the moment

$$F(\sigma,\tau) = \frac{1}{n+1} \left(\int_\sigma^s \int_\tau^t a(\xi,\eta)\, b(\xi,\eta)\, d\xi\, d\eta \right)^{n+1}, \quad x \leq \sigma \leq s,\ y \leq \tau \leq t. \tag{28}$$

We have

$$\frac{\partial^2 F}{\partial\sigma\partial\tau}(\sigma,\tau) = a(\sigma,\tau)\, b(\sigma,\tau) + \left(\int_\sigma^s \int_\tau^t a(\xi,\eta)\, b(\xi,\eta)\, d\xi\, d\eta \right)^n + \text{ nonnegative term}. \tag{29}$$

Integrating with respect to σ and τ the preceding equality on the rectangle $x \leq \sigma \leq s$, $y \leq \tau \leq t$, we obtain that

$$I \leq F(s,t) - F(s,y) - F(x,t) + F(x,y) = F(x,y), \tag{30}$$

because $F(s,t) = F(s,y) = F(x,t) = 0$. It follows from (28) and (30) that (27) is true and, using (26), we finally deduce

$$k_{n+1}(x,y,s,t) \leq \frac{1}{(n+1)!} a(x,y)\, b(s,t) \left(\int_x^s \int_y^t a(\sigma,\tau)\, b(\sigma,\tau)\, d\sigma\, d\tau \right)^{n+1}, \tag{31}$$

which implies that the inequality (24) is true for $n \geq 1$ and $(x,y,s,t) \in \Delta_2$. Then, from (23) we easily obtain

$$r(x,y,s,t) \leq a(x,y)\, b(s,t) \exp\left(\int_x^s \int_y^t a(\sigma,\tau)\, b(\sigma,\tau)\, d\sigma\, d\tau \right) \tag{32}$$

for $(x,y,s,t) \in \Delta_2$, if $k(x,y,s,t) = a(x,y)\, b(s,t)$. Thus, we can state

Proposition 3. *Assume that*

$$u(x,y) \leq f(x,y) + a(x,y) \int_x^\alpha \int_y^\beta b(s,t)\, u(s,t)\, ds\, dt \tag{33}$$

where all involved functions are continuous and nonnegative on Δ_1. Then we have

$$u(x,y) \le f(x,y) + a(x,y) \int_x^\alpha \int_y^\beta b(s,t) \exp\left(\int_x^s \int_y^t a(\sigma,\tau) b(\sigma,\tau)\, d\sigma\, d\tau \right) f(s,t)\, ds\, dt \quad (34)$$

for $(x,y) \in \Delta_1$.

Corollary 3. *Assume that the hypotheses of Proposition 3 are satisfied and that $f = f(x,y)$ is nonincreasing with respect to each of its variables. Then it follows*

$$u(x,y) \le f(x,y) \left((1 + a(x,y)) \int_x^\alpha \int_y^\beta b(s,t) \exp\left(\int_x^s \int_y^t a(\sigma,\tau) b(\sigma,\tau)\, d\sigma d\tau \right) ds\, dt \right) \quad (35)$$

for $(x,y) \in \Delta_1$.

Corollary 4. *Assume that the hypotheses of Proposition 3 are satisfied and that $f = f(x,y)$ is nonincreasing, $a = a(x,y)$ is nondecreasing. Then it follows*

$$u(x,y) \le f(x,y) \exp\left(\int_x^\alpha \int_y^\beta a(s,t) b(s,t)\, ds\, dt, \right) \quad (36)$$

for $(x,y) \in \Delta_1$.

Proof. Using the upper bounding (35) and the inequality $a(x,y) \le a(s,t)$ for $x \le s \le \alpha$, $y \le t \le \beta$, we have only to show that

$$\begin{aligned} 1 + \int_x^\alpha \int_y^\beta a(s,t) b(s,t) \exp\left(\int_x^s \int_y^t a(\sigma,\tau) b(\sigma,\tau)\, d\sigma\, d\tau \right) dsdt \le \\ \le \exp\left(\int_x^\alpha \int_y^\beta a(s,t) b(s,t)\, ds\, dt \right) \end{aligned} \quad (37)$$

for $(x,y) \in \Delta_1$. For (x,y) fixed, we define the function

$$F(s,t) = \exp\left(\int_x^s \int_y^t a(\sigma,\tau) b(\sigma,\tau)\, d\sigma\, d\tau \right) \quad (38)$$

for $x \le s \le \alpha$, $y \le t \le \beta$. Integrating the inequality

$$a(s,t) b(s,t) F(s,t) \le \frac{\partial^2 F}{\partial s \partial t}(s,t) \quad (39)$$

on the rectangle $x \le s \le \alpha$, $y \le t \le \beta$, we obtain

$$\int_x^\alpha \int_y^\beta a(s,t) b(s,t) F(s,t)\, ds\, dt \le F(\alpha,\beta) - F(\alpha,y) - F(x,\beta) + F(x,y) \quad (40)$$

where $F(\alpha, y) = F(x, \beta) = F(x, y) = 1$ and $F(\alpha, \beta)$ is equal to the right-hand side of (37). Consequently, (37) is true and this completes the proof. ■

Remark 2. If, in (33), we take $f(x, y) = K = const > 0$ and $a(x, y) = 1$, we find an example in which the bound given by using the resolvent kernel, namely

$$u(x,y) \le K\left(1 + \int_x^\alpha \int_y^\beta b(s,t)\exp\left(\int_x^s \int_y^t b(\sigma,\tau)\,d\sigma\,d\tau\right) ds\,dt\right), \quad (x,y) \in \Delta_1 \tag{41}$$

is better than those obtained by the Bellman-Bihari procedure, namely

$$u(x,y) \le K \exp\left(\int_x^\alpha \int_y^\beta b(s,t)\,ds\,dt\right), \quad (x,y) \in \Delta_1. \tag{42}$$

Of course, it was supposed that $b = b(x, y) \ge 0$ is continuous on Δ_1.

Application. (A problem of the Darboux-type). Let us consider the integro-partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = a(x,y)\,u(x,y) + \int_x^\alpha \int_y^\beta b(x,y,s,t)\,u(s,t)\,ds\,dt, \quad (x,y) \in \Delta_1 \tag{43}$$

under the boundary value conditions

$$u(x,\beta) = \varphi(x), \; u(\alpha,y) = \psi(y) \qquad (\varphi(\alpha) = \psi(\beta)) \tag{44}$$

where $a = a(x, y)$ is continuous on Δ_1, $b = b(x, y, s, t)$ is continuous on Δ_2 and the functions $\varphi = \varphi(x)$, $\psi = \psi(y)$ have continuous derivatives of the first order for $\alpha_1 \le x \le \alpha$, respectively $\beta_1 \le y \le \beta$. The existence and the uniqueness of the solution for this problem, in the class of functions having continuous partial deriavatives $\partial u/\partial x$, $\partial u/\partial y$ and $\partial^2 u/\partial x\partial y = \partial^2 u/\partial y\partial x$, may be proved by the method of successive approximations, for the equivalent integral equation

$$u(x,y) = f(x,y) + \int_x^\alpha \int_y^\beta a(s,t)\,u(s,t)\,ds\,dt + \int_x^\alpha \int_y^\beta \left(\int_s^\alpha \int_t^\beta b(s,t,\sigma,\tau)\,u(\sigma,\tau)\,d\sigma\,d\tau\right) ds\,dt \tag{45}$$

for $(x, y) \in \Delta_1$. Here, $f(x, y)$ stands for $\varphi(x)+\psi(y)-\psi(\beta)$. Putting $K = \sup|f(x, y)| < \infty$ for $(x, y) \in \Delta_1$ and using Corollary 2, we obtain for the solution of the problem (43) – (44) the bound

$$|u(x,y)| \le K \exp\left(\int_x^\alpha \int_y^\beta |a(s,t)|\,ds\,dt + \int_x^\alpha \int_y^\beta \left(\int_s^\alpha \int_t^\beta |b(s,t,\sigma,\tau)|\,d\sigma\,d\tau\right) ds\,dt\right) \tag{46}$$

for $(x, y) \in \Delta_1$.

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