

A Uniqueness Theorem for Radiative Transfer Equation

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Abstract. A uniqueness theorem of solution of the radiative transfer equation for a medium with nonvanishing absorption and coherent scattering interactions, for given source function, initial and boundary conditions is derived. Time-independent and stationary problems are analyzed. Relation of time-independent case with the stationary problem is deduced.

1 Introduction

To study the physical conditions of stars and other astrophysical objects we are almost exclusively restricted to an analysis of the photons that escape from the medium. It is therefore necessary to study the processes of emission, absorption, scattering and radiative transfer in order to make proper inferences from observations of radiation.

Mathematically, one is confronted with the task of seeking solution to initial and boundary value problems of an integro-differential equation for transfer or transport processes. Exact solutions are rarely available in most cases. Consequently, a wide variety of methods have developed which give approximate solutions of such problems. The methods for solving transfer problems are in an advanced state of development. Several excellent treatise are available in the field dealing with transfer problems in stellar atmospheres [1], [2], radiative heat transfer in curved media [3], and neutron transport [5]. An important literature has been devoted to the inverse problems in transport theory (see references from [4]).

In this paper, we proof the uniqueness of solution of radiative transfer equation for a medium with coherent scattering, for given source, initial and boundary conditions. In Section 2 we write the radiative transfer equation for coherent scattering. In Section 3 we formulate and proof the uniqueness theorem for time-dependent transfer equation. Section 4 is devoted to the stationary problem. In Section 5 we show how, in certain conditions, a time-dependent problem can be formally reduced to a time-independent one.

2 The radiative transfer equation

The energy of radiation field is carried by point, massless particles called photons. With each photon we associate a frequency ν such that the energy of the photon is $h\nu$, where h is Planck's constant. Between collisions with matter, a photon is supposed to travel in a straight line with speed $c\mathbf{\Omega}$, where c is the vacuum speed of light and $\mathbf{\Omega}$ is the direction of travel of the photon (a unit vector).

The equation of transfer is the mathematical statement of the conservation of photons. This is an integro-differential equation and may be written, suppressing the t and \mathbf{x} dependences, in the form [6]

$$\begin{aligned} \frac{\partial f(\nu, \mathbf{\Omega})}{\partial t} + c\mathbf{\Omega} \cdot \nabla f(\nu, \mathbf{\Omega}) + c\sigma(\nu) f(\nu, \mathbf{\Omega}) = \\ = q(\nu) + c \int_0^\infty d\nu' \int_{\Omega} \sigma_s(\nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}) f(\nu, \mathbf{\Omega}') d\Omega', \end{aligned} \quad (2.1)$$

where $f(t, \mathbf{x}, \nu, \mathbf{\Omega})$ is the distribution function of photons, $q(t, \mathbf{x}, \nu)$ is the source function, $\sigma_s(t, \mathbf{x}, \nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega})$ is the differential scattering coefficient, and $\sigma(t, \mathbf{x}, \nu)$ is the total interaction coefficient, the sum of the absorption coefficient $\sigma_a(t, \mathbf{x}, \nu)$ and the scattering coefficient $\sigma_s(t, \mathbf{x}, \nu)$ given by

$$\sigma_s(\nu') = \int_0^\infty d\nu \int_{\Omega} \sigma_s(\nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}) d\Omega = 2\pi \int_0^\infty d\nu \int_{-1}^1 \sigma_s(\nu' \rightarrow \nu, \mu) d\mu. \quad (2.2)$$

The last equality follows by choosing $\mathbf{\Omega}'$ as direction of axis for the Ω integration and letting $\mu = \mathbf{\Omega}' \cdot \mathbf{\Omega}$. Often it is convenient to decompose the differential scattering coefficient into the product of two terms according to [7]

$$\sigma_s(\nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}) = \sigma_s(\nu') K(\nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}), \quad (2.3)$$

where kernel $K(\nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega})$ is properly termed probability distribution in that it is normalized to unity

$$\int_0^\infty d\nu \int_{\Omega} K(\nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}) d\Omega = 1, \quad (2.4)$$

as follows from Eqs. (2.2) and (2.3).

Let $\omega(t, \mathbf{x}, \nu)$ denote the probability of scattering given that a collision has occurred, i.e.,

$$\omega(\nu) = \frac{\sigma_s(\nu)}{\sigma_a(\nu) + \sigma_s(\nu)} = \frac{\sigma_s(\nu)}{\sigma(\nu)}. \quad (2.5)$$

For the case of no absorption $\sigma_a(\nu) = 0$, so that $\omega(\nu) = 1$. This case is referred to as the *conservative* or *perfect scattering* case. For the case of no scattering $\sigma_s(\nu) = 0$, so that $\omega(\nu) = 0$. Otherwise

$$0 < \omega(\nu) < 1. \quad (2.6)$$

By virtue of Eqs. (2.2), (2.3), and (2.5), the radiative transfer equation becomes

$$\frac{\partial f(\nu, \mathbf{\Omega})}{\partial t} + c\mathbf{\Omega} \cdot \nabla f(\nu, \mathbf{\Omega}) + c\sigma(\nu) f(\nu, \mathbf{\Omega}) =$$

$$= q(\nu) + c \int_0^\infty d\nu' \int_\Omega \sigma_s(\nu') K(\nu' \rightarrow \nu, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) f(\nu, \boldsymbol{\Omega}') d\Omega'. \quad (2.7)$$

If there is no frequency change upon scattering, the kernel K contains a Dirac function in frequency

$$K(\nu' \rightarrow \nu, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) = K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) \delta(\nu' - \nu)$$

and the scattering is said to be *coherent*. It can be seen from (2.4) that $K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega})$ is normalized according to

$$\int_\Omega K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) d\Omega = 1.$$

In the case of the coherent scattering, taken into account ((2.5)), Eq. ((2.7)) can be written

$$\begin{aligned} \frac{\partial f(\nu, \boldsymbol{\Omega})}{\partial t} + c\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} f(\nu, \boldsymbol{\Omega}) + c\sigma(\nu) f(\nu, \boldsymbol{\Omega}) = \\ = q(\nu) + c\omega(\nu)\sigma(\nu) \int_0^\infty d\nu' \int_\Omega K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) f(\nu, \boldsymbol{\Omega}') d\Omega'. \end{aligned} \quad (2.8)$$

In this equation, the frequency ν enters only as a trivial parameter, since photons of different frequency are completely uncoupled.

3 Uniqueness theorem

In solving the radiative transfer equation, it is apparently necessary to specify both the distribution function at some initial time $t = 0$, and the source function throughout all space. One can then obtain a complete solution. In a practical case, however, one is interested only in the distribution in a restricted domain \mathcal{V} of space, bounded by a surface \mathcal{S} . We assume this domain to be non-re-entrant. By non-re-entrant we mean that any photon which leaves the domain will not re-enter it through another part of its surface. If the domain is re-entrant, we enclose it in a non-re-entrant hypothetical domain bounded by an imposed, rather than real, surface. The new domain is then non-re-entrant, but consists, in part, of vacuum ($\sigma = 0$, $q = 0$).

Since the equation (2.8) is a first order equation in time and space, initial and boundary conditions are required. We need to specify the initial condition

$$f(0, \mathbf{x}, \nu, \boldsymbol{\Omega}) = f_0(\mathbf{x}, \nu, \boldsymbol{\Omega}), \quad \mathbf{x} \in \mathcal{V}, \quad (3.1)$$

and the distribution function at all points on the surface \mathcal{S} in the incoming direction, which implies the boundary condition

$$f(t, \mathbf{x}, \nu, \boldsymbol{\Omega})|_{\mathbf{x}=\mathbf{x}_s} = f_s(t, \mathbf{x}_s, \nu, \boldsymbol{\Omega}), \quad t \in [0, \infty), \quad \mathbf{n} \cdot \boldsymbol{\Omega} < 0, \quad (3.2)$$

where f_0 and f_s are specified functions of their arguments, \mathbf{x}_s is a point on the surface \mathcal{S} , and \mathbf{n} is an outward normal vector to \mathcal{S} in this point.

Theorem 1. *The distribution function $f \in C^1([0, \infty) \cup \mathcal{V})$ is uniquely determined by*

- (i) *the source function q within \mathcal{V} ,*
- (ii) *the initial distribution f_0 in \mathcal{V} , and*
- (iii) *the distribution f_s incident on \mathcal{S} .*

PROOF. Let f_1 and f_2 be two solutions of the radiative transfer equation (2.8) corresponding to the same source q and the same initial and boundary conditions (3.1) and (3.2). Writing $F = f_1 - f_2$, we see that in \mathcal{V} , F obeys the following equation

$$\begin{aligned} \frac{\partial F(\nu, \boldsymbol{\Omega})}{\partial t} + c\boldsymbol{\Omega} \cdot \nabla F(\nu, \boldsymbol{\Omega}) + c\sigma(\nu) F(\nu, \boldsymbol{\Omega}) = \\ = c\omega(\nu)\sigma(\nu) \int_0^\infty d\nu' \int_\Omega K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) F(\nu, \boldsymbol{\Omega}') d\Omega', \end{aligned} \quad (3.3)$$

with the initial condition

$$F(0, \mathbf{x}, \nu, \boldsymbol{\Omega}) = 0, \quad \mathbf{x} \in \mathcal{V}, \quad (3.4)$$

and the boundary condition

$$F(t, \mathbf{x}, \nu, \boldsymbol{\Omega})|_{\mathbf{x}=\mathbf{x}_s} = 0, \quad t \in [0, \infty), \quad \mathbf{n} \cdot \boldsymbol{\Omega} < 0. \quad (3.5)$$

We shall prove that $F \equiv 0$.

By multiplying Eq. (3.3) by $F(\nu, \boldsymbol{\Omega})$ and integrating over t from 0 to t_0 , $t_0 > 0$, over \mathcal{V} , and Ω , applying the Gauss's theorem to the second term, we obtain

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{V}} dV \int_\Omega [F^2(t_0, \mathbf{x}, \nu, \boldsymbol{\Omega}) - F^2(0, \mathbf{x}, \nu, \boldsymbol{\Omega})] d\Omega + \\ + \frac{c}{2} \int_0^{t_0} dt \int_S dS \int_\Omega (\mathbf{n} \cdot \boldsymbol{\Omega}) F^2(\nu, \boldsymbol{\Omega}) d\Omega + c \int_0^{t_0} dt \int_{\mathcal{V}} dV \int_\Omega \sigma(\nu) F^2(\nu, \boldsymbol{\Omega}) d\Omega - \\ - c \int_0^{t_0} dt \int_{\mathcal{V}} dV \int_\Omega d\Omega \int_\Omega \omega(\nu)\sigma(\nu) K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) F(\nu, \boldsymbol{\Omega}') F(\nu, \boldsymbol{\Omega}) d\Omega' = 0, \end{aligned} \quad (3.6)$$

where, in the last three terms the t and \mathbf{x} dependences were suppressed. The first term is non-negative, the second term is null by virtue of condition (3.4). The third term is non-negative, since $F(t, \mathbf{x}, \nu, \boldsymbol{\Omega})$ vanishes unless $\mathbf{n} \cdot \boldsymbol{\Omega}$ is positive. In order to place limits on the last term, we consider the inequality

$$\int_\Omega d\Omega \int_\Omega K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) [F^2(\nu, \boldsymbol{\Omega}') - F^2(\nu, \boldsymbol{\Omega})]^2 d\Omega' \geq 0.$$

Expanding the square and noting that $K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega})$ is normalized such that

$$\int_\Omega K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) d\Omega' = \int_\Omega K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) d\Omega = 1,$$

we can write the previous inequality as

$$\int_\Omega d\Omega \int_\Omega K(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) F(\nu, \boldsymbol{\Omega}') F(\nu, \boldsymbol{\Omega}) d\Omega' \leq \int_\Omega F^2(\nu, \boldsymbol{\Omega}) d\Omega.$$

Consequently, the last two terms from Eq. (3.6) can be increased by

$$c \int_0^{t_0} dt \int_{\mathcal{V}} dV \int_\Omega [1 - \omega(\nu)] \sigma(\nu) F^2(\nu, \boldsymbol{\Omega}) d\Omega,$$

which, in virtue of (2.6), is non-negative. Thus, the left-hand side of Eq. (3.6) can be increased by a sum of two non-negative terms. It follows that

$$F(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) = 0, \quad t \in [0, \infty), \quad \mathbf{x} \in \mathcal{V},$$

for all directions Ω , which proves the uniqueness theorem. ■

4 Uniqueness of the stationary problem

In certain cases, we may seek solutions of the radiative transfer equation which are constant in time in a domain \mathcal{V} of space. For example, if the source function q within \mathcal{V} and the distribution function f_s on \mathcal{S} are time-independent, then we might expect to be able to find solutions to the time-independent transfer equation

$$\begin{aligned} c\mathbf{\Omega} \cdot \nabla f(\mathbf{x}, \nu, \mathbf{\Omega}) + c\sigma(\mathbf{x}, \nu) f(\mathbf{x}, \nu, \mathbf{\Omega}) &= \\ = q(\mathbf{x}, \nu) + c\omega(\mathbf{x}, \nu) \sigma(\mathbf{x}, \nu) \int_0^\infty d\nu' \int_{\Omega} K(\mathbf{x}, \mathbf{\Omega}' \cdot \mathbf{\Omega}) f(\mathbf{x}, \nu, \mathbf{\Omega}') d\Omega', \end{aligned} \quad (4.1)$$

with the boundary condition

$$f(\mathbf{x}, \nu, \mathbf{\Omega})|_{\mathbf{x}=\mathbf{x}_s} = f_s(\mathbf{x}, \nu, \mathbf{\Omega}), \quad \mathbf{n} \cdot \mathbf{\Omega} < 0, \quad (4.2)$$

Theorem 2. *The stationary distribution function $f \in C^1(\mathcal{V})$ is uniquely determined by*

- (i) *the stationary source function q within \mathcal{V} , and*
- (ii) *the stationary distribution function f_s on \mathcal{S} .*

The proof of this theorem proceeds exactly as did the proof of uniqueness in the previous section, except that the integration over t is omitted. Noting with $F = f_1 - f_2$ the difference of two solutions of Eq. (4.1) with the condition (4.2), we have

$$\begin{aligned} c\mathbf{\Omega} \cdot \nabla F(\mathbf{x}, \nu, \mathbf{\Omega}) + c\sigma(\mathbf{x}, \nu) F(\mathbf{x}, \nu, \mathbf{\Omega}) &= \\ = c\omega(\mathbf{x}, \nu) \sigma(\mathbf{x}, \nu) \int_0^\infty d\nu' \int_{\Omega} K(\mathbf{x}, \mathbf{\Omega}' \cdot \mathbf{\Omega}) F(\mathbf{x}, \nu, \mathbf{\Omega}') d\Omega', \end{aligned} \quad (4.3)$$

with the boundary condition

$$F(\mathbf{x}, \nu, \mathbf{\Omega})|_{\mathbf{x}=\mathbf{x}_s} = 0, \quad \mathbf{n} \cdot \mathbf{\Omega} < 0. \quad (4.4)$$

By multiplying Eq. (4.3) by $F(\mathbf{x}, \nu, \mathbf{\Omega})$ and integrating over \mathcal{V} , and Ω , applying the Gauss's theorem to the first term, we obtain

$$\begin{aligned} \frac{c}{2} \int_{\mathcal{S}} dS \int_{\Omega} (\mathbf{n} \cdot \mathbf{\Omega}) F^2(\mathbf{x}, \nu, \mathbf{\Omega}) d\Omega + c \int_{\mathcal{V}} dV \int_{\Omega} \sigma(\mathbf{x}, \nu) F^2(\mathbf{x}, \nu, \mathbf{\Omega}) d\Omega - \\ - c \int_{\mathcal{V}} dV \int_{\Omega} d\Omega \int_{\Omega} \omega(\mathbf{x}, \nu) \sigma(\mathbf{x}, \nu) K(\mathbf{x}, \mathbf{\Omega}' \cdot \mathbf{\Omega}) F(\mathbf{x}, \nu, \mathbf{\Omega}') F(\mathbf{x}, \nu, \mathbf{\Omega}) d\Omega' = 0. \end{aligned}$$

By applying the same trick as was used in the previous section, we are able to increase the last two terms of this equation by

$$c \int_{\mathcal{V}} dV \int_{\Omega} [1 - \omega(\mathbf{x}, \nu)] \sigma(\mathbf{x}, \nu) F^2(\mathbf{x}, \nu, \mathbf{\Omega}) d\Omega,$$

which, in virtue of (2.6), is non-negative. It follows that

$$F(\mathbf{x}, \nu, \mathbf{\Omega}) = 0, \quad \mathbf{x} \in \mathcal{V},$$

for all directions Ω , which proves the uniqueness theorem.

5 Reduction to the stationary problem

If the medium properties are such that the absorption and scattering interactions are time-independent, i.e. σ_s and σ are constant in time, but the source function is time-dependent, then we can reduce the time-dependent radiative transfer equation (2.8) to an equation which formally looks like the stationary equation. This may be done by defining the Laplace transforms of $f(t, \mathbf{x}, \nu, \boldsymbol{\Omega})$ and $q(t, \mathbf{x}, \nu)$ by

$$\begin{aligned} g(s, \mathbf{x}, \nu, \boldsymbol{\Omega}) &= \int_0^\infty f(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) e^{-st} dt, \\ p(s, \mathbf{x}, \nu) &= \int_0^\infty q(t, \mathbf{x}, \nu) e^{-st} dt. \end{aligned} \quad (5.1)$$

Multiplying Eq. (2.8) by e^{-st} and integrating over t from 0 to ∞ , we find

$$\begin{aligned} sg(s, \mathbf{x}, \nu, \boldsymbol{\Omega}) - f_0(s, \mathbf{x}, \nu, \boldsymbol{\Omega}) + c\boldsymbol{\Omega} \cdot \nabla g(s, \mathbf{x}, \nu, \boldsymbol{\Omega}) + c\sigma(\mathbf{x}, \nu)g(s, \mathbf{x}, \nu, \boldsymbol{\Omega}) = \\ p(s, \mathbf{x}, \nu) + c\omega(\mathbf{x}, \nu)\sigma(\mathbf{x}, \nu) \int_0^\infty d\nu' \int_\Omega K(\mathbf{x}, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) g(s, \mathbf{x}, \nu, \boldsymbol{\Omega}') d\Omega'. \end{aligned}$$

Rearranging terms, we can write

$$\begin{aligned} c\boldsymbol{\Omega} \cdot \nabla g(\nu, \boldsymbol{\Omega}) + c\sigma'(\nu)g(\nu, \boldsymbol{\Omega}) = \\ = q'(\nu, \boldsymbol{\Omega}) + c\omega'(\nu)\sigma'(\nu) \int_0^\infty d\nu' \int_\Omega K(\mathbf{x}, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) g(\nu, \boldsymbol{\Omega}') d\Omega', \end{aligned}$$

where

$$\begin{aligned} \sigma'(s, \mathbf{x}, \nu) &= \sigma(\mathbf{x}, \nu) + \frac{s}{c} = \sigma_s(\mathbf{x}, \nu) + \left(\sigma_a(\mathbf{x}, \nu) + \frac{s}{c}\right), \\ \omega'(s, \mathbf{x}, \nu) &= \omega(\mathbf{x}, \nu) \cdot \frac{\sigma(\mathbf{x}, \nu)}{\sigma(\mathbf{x}, \nu) + \frac{s}{c}} = \frac{\sigma_s(\mathbf{x}, \nu)}{\sigma_s(\mathbf{x}, \nu) + \left(\sigma_a(\mathbf{x}, \nu) + \frac{s}{c}\right)}, \\ q'(s, \mathbf{x}, \nu, \boldsymbol{\Omega}) &= p(s, \mathbf{x}, \nu) + f_0(s, \mathbf{x}, \nu, \boldsymbol{\Omega}), \end{aligned}$$

and the boundary condition on \mathcal{S}

$$g(s, \mathbf{x}, \nu, \boldsymbol{\Omega})|_{\mathbf{x}=\mathbf{x}_s} = g_s(s, \mathbf{x}_s, \nu, \boldsymbol{\Omega}), \quad \mathbf{n} \cdot \boldsymbol{\Omega} < 0,$$

where g_s is given by (5.1) for $\mathbf{x} = \mathbf{x}_s$.

The equation for the transform g is formally identical with the stationary radiative transfer equation (4.1), with

$$\sigma, \omega, q \rightarrow \sigma', \omega', q'.$$

In particular, the changes in σ and ω are just those which would occur if, keeping all other coefficients unchanged, we increase the absorption coefficient σ_a by an amount s/c .

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