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A New Class of Holomorphic Functions Defined by Sălăgean Differential Operator

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Abstract. By using the differential operator $D^n f(z)$, $z \in U$, given by Definition 1, we introduce a class of holomorphic functions, denoted by $M_n(\alpha)$, and we obtain some differential subordinations.

1 Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}[U]$ be the space of holomorphic function in U.

We let:

$$A_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$.

We let $\mathcal{H}[a, n]$ denote the class of analytic functions in U of the form

$$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \quad z \in U.$$

Let

$$K = \left\{f \in A : \ \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U\right\}$$

denote the class of normalized convex functions in U.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We use the following subordination results.

Lemma A. (Hallenbeck and Ruscheweyh [1, p. 71]) *Let h be a convex function with $h(0) \equiv a$ and let $\gamma \in \mathbb{C}^*$ be a complex with* $\mathrm{Re}\,\gamma \geq 0$. *If $p \in \mathcal{H}[U]$ with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma-1} dt.$$

The function q is convex and is the best dominant.

(The definition of best dominant is given in [2].)

Definition 1. (G. S. Sălăgean [4]) For $f \in A$ and $n \geq 0$ we define the operator $D^n f$ by

$$D^0 f(z) = f(z)$$

$$D^{n+1} f(z) = z[D^n f(z)]', \ z \in U.$$

2 Main results

Definition 2. If $\alpha < 1$ and $n \in \mathbb{N}$, let $M_n(\alpha)$ denote the class of functions $f \in A$ which satisfy the inequality:

$$\text{Re } [D^n f(z)]' > \alpha.$$

Theorem 1. *Let $h \in \mathcal{H}[U]$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\text{Re} \left[1 + \frac{zh''(z)}{h'(z)} \right] > -\frac{1}{2}.$$

If $f \in A$ and verifies the differential subordination

$$[D^{n+1} f(z)]' \prec h(z), \quad z \in U, \tag{1}$$

then

$$[D^n f(z)]' \prec g(z)$$

where

$$g(z) = \frac{1}{z} \int_0^z h(t) t dt.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [1,2] shows that the function g is convex. From

$$D^{n+1} f(z) = z[D^n f(z)]', \quad z \in U,$$

we obtain

$$[D^{n+1} f(z)]' = [D^n f(z)]' + z[D^n f(z)]'', \quad z \in U.$$

If we let $p(z) = [D^n f(z)]'$, $p'(z) = [D^n f(z)]''$, then we obtain

$$[D^{n+1} f(z)]' = p(z) + zp'(z)$$

and (1) becomes

$$p(z) + zp'(z) \prec h(z).$$

By using Lemma A, we have

$$p(z) \prec g(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function g is convex and is the best dominant. □

Theorem 2. *Let $h \in \mathcal{H}[U]$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}.$$

If $f \in A$ verifies the differential subordination

$$[D^n f(z)]' \prec h(z), \quad z \in U, \tag{2}$$

then

$$\frac{D^n f(z)}{z} \prec g(z)$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)tdt.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [1,2] shows that the function g is convex.

We let

$$p(z) = \frac{D^n f(z)}{z}, \quad z \in U,\ z \neq 0,$$

and we obtain

$$D^n f(z) = zp(z).$$

By differentiating we obtain

$$[D^n f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (2) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

By using Lemma A we have

$$p(z) \prec q(z)$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt,$$

where q is convex and is the best dominant. □

Corollary 1. *If $f \in M_n(\alpha)$, then*

$$\operatorname{Re}\frac{D^n f(z)}{z} > 2\alpha - 1 + 2(1-\alpha)\ln 2.$$

Proof. From Theorem 2 we deduce

$$\frac{D^n f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt,$$

where

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}, \quad z \in U.$$

Hence

$$\text{Re}\,\frac{D^n f(z)}{z} > \text{Re}\,g(1) = 2\alpha - 1 + 2(1-\alpha)\ln 2.$$

□

We remark that in the case of meromorphic functions a similar result was obtained by M. Pap in [3].

References

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[3] M. Pap, *On certain subclasses of meromorphic m-valent close-to-convex functions*, P.U.M.A., vol. 9(1998), No. 1-2, 155-163.

[4] G. Şt. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, 1013(1983), 362-372.