

Briot-Bouquet Differential Superordinations and Sandwich Theorem

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Abstract. Let $f \in A$, $\beta > 0$, $\gamma \geq -\beta$. We consider the following integral operator

$$F(z) = \left[\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma-1} dt \right]^{\frac{1}{\beta}}, \quad z \in U. \quad (1)$$

By using this integral operator we obtain a Briot-Bouquet differential superordination and sandwich theorem.

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1 Introduction

Let $\mathcal{H}(U)$ denote the class of functions analytic in the unit disc

$$U = \{z \in \mathbb{C}, |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U\},$$

and $A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1} z^{n+1} + \dots, z \in U\}$ with $A_1 = A$.

A function $f \in \mathcal{H}[a, 1]$ is convex in U if it is univalent and $f(U)$ is convex. It is well known that f is convex if and only if $f'(0) \neq 0$ and

$$\operatorname{Re} \frac{z f''(z)}{f'(z)} + 1 > 0, \quad z \in U.$$

Let Q denote the set of functions f that are analytic and injective on the set $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U, \lim_{z \rightarrow \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Many of the inclusion results that follow can be written very easily in terms of subordination and superordination. We recall these definitions. Let $f, F \in \mathcal{H}(U)$ and let F be univalent in U . The function F is said to be superordinate to f , or f is subordinate to F , written $f \prec F$, if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let β and γ be complex numbers. Let Ω_2 and Δ_2 be sets in the complex plane, and let p be analytic in the unit disc U . In a series of articles the authors [1, p. 80-119] have determined conditions so that

$$\left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \mid z \in U \right\} \subset \Omega_2 \Rightarrow p(U) \subset \Delta_2. \quad (2)$$

The differential operator on the left is known as the Briot-Bouquet differential operator. The main concern in this subject is to find the smallest set Δ_2 in \mathbb{C} for which (2) holds.

In [4] the authors consider the dual problem of determining conditions so that

$$\Omega_1 \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \mid z \in U \right\} \Rightarrow \Delta_1 \subset p(U). \quad (3)$$

In particular we are interested in determining the largest set Δ_1 in \mathbb{C} for which (3) holds.

If the sets Ω and Δ in (2) and (3) are simply connected domains not equal to \mathbb{C} , then it is possible to rephrase these expressions very neatly in terms of subordination and superordination in the forms:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \Rightarrow p(z) \prec q_2(z) \quad (2')$$

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q_1(z) \prec p(z). \quad (3')$$

The left side of (2') is called a Briot-Bouquet differential subordination, and the function q_2 is called a dominant of the differential subordination. The best dominant which provides a sharp result, is the dominant that is subordinate to all other dominants.

In a recent paper [3] the authors introduced the dual concept of a differential superordination. In light of those results we call the left side of (3') a Briot-Bouquet differential superordination, and the function q_1 is called a subordinant of the differential superordination. The best subordinant, which provides a sharp result is the subordinant which is superordinate to all other subordinants.

In [4] the authors combine (2') and (3') and obtain a condition so that the Briot-Bouquet sandwich

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \quad (4)$$

implies that $q_1(z) \prec p(z) \prec q_2(z)$.

In order to prove the new results we shall use the following lemma:

Lemma A. [4, Corollary 1.1] *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U , with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$$

has a univalent solution q that satisfies $q(0) = a$ and $q(z) \prec h(z)$. If $p \in \mathcal{H}[a, 1] \cap Q$ and $p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}$ is univalent in U , then

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}$$

implies

$$q(z) \prec p(z).$$

The function q is the best subordinated.

Lemma B. [1, Th. 3.2.b, p. 83] Let h be a convex function in U , with $h(0) = a$ and let n be a positive integer. Suppose that the Briot-Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{q(z) + 1} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$.

If $p \in \mathcal{H}[a, n]$ satisfies

$$p(z) + \frac{zp'(z)}{p(z) + 1} \prec h(z)$$

then $p(z) \prec q(z)$ and q is the best (a, n) dominant.

2 Main Results

Theorem 1. Let $R \in (0, 1]$ and let h be convex in U , with $h(0) = 1$, defined by

$$h(z) = 1 + Rz + \frac{zR}{2 + Rz}, \quad z \in U.$$

If $f \in A$ and $\frac{zf'(z)}{f(z)}$ is univalent, $\frac{zF'(z)}{F(z)} \in \mathcal{H}[1, 1] \cap Q$ and

$$h(z) \prec \frac{zf'(z)}{f(z)}, \quad z \in U \tag{5}$$

then

$$q(z) = 1 + Rz \prec \frac{zF'(z)}{F(z)}, \quad z \in U,$$

where F is given by (1).

The function q is the best subordinated.

Proof. In [5] the authors have showed that

$$h(z) = 1 + Rz + \frac{zR}{2 + Rz}, \quad R \in (0, 1] \tag{6}$$

is convex, and $q(z) = 1 + Rz$ is a univalent solution of (3) which satisfies $q(0) = 1$ and $q(z) \prec h(z)$, $z \in U$.

From (1) we have

$$z^\gamma F^\beta(z) = (\beta + \gamma) \int_0^z f^\beta(t) t^{\gamma-1} dt, \quad z \in U.$$

By using the derivative of this equality, with respect to z , after a short calculation, we obtain

$$\gamma F^\beta(z) + z\beta F^{\beta-1}(z)F'(z) = (\beta + \gamma)f^\beta(z).$$

This equality is equivalent to

$$F^\beta(z) \left[\gamma + z\beta \frac{F'(z)}{F(z)} \right] = (\beta + \gamma)f^\beta(z), \quad z \in U. \quad (7)$$

If we let

$$p(z) = \frac{zF'(z)}{F(z)}, \quad z \in U, \quad (8)$$

then (7) becomes

$$F^\beta(z)[\gamma + \beta p(z)] = (\beta + \gamma)f^\beta(z), \quad z \in U. \quad (9)$$

By using the derivative of (9) with respect to z , after a short calculation, we obtain

$$\frac{zF'(z)}{F(z)} + \frac{zp'(z)}{\gamma + \beta p(z)} = \frac{zf'(z)}{f(z)}$$

which, using (8), is equivalent to

$$p(z) + \frac{zp'(z)}{\gamma + \beta p(z)} = \frac{zf'(z)}{f(z)}, \quad z \in U.$$

Using (5) we have

$$1 + Rz + \frac{Rz}{2 + Rz} \prec p(z) + \frac{zp'(z)}{\gamma + \beta p(z)}, \quad z \in U.$$

By using Lemma A we deduce that

$$q(z) \prec p(z) = \frac{zF'(z)}{F(z)}, \quad 1 + Rz \prec \frac{zF'(z)}{F(z)}, \quad z \in U.$$

The function q is the best subordinant.

Theorem 2. Let h be convex in U , with $h(0) = 1$, defined by

$$h(z) = 1 + z + \frac{z}{z+2}, \quad z \in U.$$

If $f \in A$ and

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in U \quad (10)$$

then

$$\frac{zF'(z)}{F(z)} \prec 1 + z,$$

where F is given by (1). The function $q(z) = 1 + z$ is best dominant.

Proof. In [5] the authors have showed that

$$h(z) = 1 + z + \frac{z}{z+2}$$

is convex.

From (1) we have

$$z^\gamma F^\beta(z) = (\beta + \gamma) \int_0^z f^\beta(t) t^{\gamma-1} dt, \quad z \in U.$$

Following the steps from the proof of Theorem 1 we obtain:

$$p(z) + \frac{zp'(z)}{\gamma + \beta p(z)} = \frac{zf'(z)}{f(z)}.$$

Using (10) we have

$$p(z) + \frac{zp'(z)}{\gamma + \beta p(z)} \prec h(z).$$

By applying Lemma B we obtain

$$p(z) = \frac{zF'(z)}{F(z)} \prec q(z) = 1 + z, \quad z \in U.$$

The function $q(z) = 1 + z$ is the best dominant.

Using the conditions from Theorem 1 and Theorem 2 we can write the following

Corollary. *If $f \in A$ and*

$$1 + Rz + \frac{zR}{2 + Rz} \prec \frac{zf'(z)}{f(z)} \prec 1 + z + \frac{z}{2 + z}, \quad z \in U$$

then

$$1 + Rz \prec \frac{zF'(z)}{F(z)} \prec 1 + z, \quad z \in U,$$

where $R \in (0, 1]$

References

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