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The Strengthening of Minkowski Inequality for Mixed Projection Bodies

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Abstract. The purpose of present paper is to establish a stengthened form of the Minkowski inequality for mixed projection bodies.

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1 Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n(n > 2)$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}.

We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K, \cdot) : S^{n-1} \to R$, denote the support function of $K \in \mathcal{K}^n$; i.e.

$$h(K, u) = Max\{u \cdot x : x \in K\}, \quad u \in S^{n-1}, \tag{1}$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

where $| \cdot |_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

For a convex body K and a nonnegative scalar $\lambda, \lambda K$ is used to denote $\{\lambda x : x \in K\}$. For $K_i \in \mathcal{K}^n, \lambda_i \geq 0, (i = 1, 2, \ldots, r)$, *the Minkowski linear combination* $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in \mathcal{K}^n : x_i \in K_i\}. \tag{2}$$

If $K_i \in \mathcal{K}^n(i = 1, 2, \ldots, r)$ and $\lambda_i(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental impotence is the fact that the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial in λ_i given by $[1, p.275]$

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{3}$$

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where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (3), it is called *the mixed volume* of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is usually written $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the ith projection measure (Quermassintegral) of K and is written as $W_i(K)$.

If $K_1, \ldots, K_r \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, then the projection body of the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathcal{K}^n$ can be written as a symmetric homogeneous polynomial of degree $(n-1)$ in the λ_i [2]:

$$\Pi(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \lambda_{i_1} \ldots \lambda_{i_{n-1}} \Pi_{i_1 \cdots i_{n-1}} \tag{4}$$

where the sum is a Minkowski sum taken over all $(n-1)$-tuples $(i_1, \ldots, i_{n-1})$ of positive integers not exceeding r. The body $\Pi_{i_1 \ldots i_{n-1}}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_{n-1}}$, and is uniquely determined by (4), it is called *the mixed projection bodies* of $K_{i_1}, \ldots, K_{i_{n-1}}$, and is written as $\Pi(K_i, \ldots, K_{i_{n-1}})$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_{i_1}, \ldots, K_{i_{n-1}})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, B)$ is called the ith projection body of K and is denoted $\Pi_i K$.

2 Main Results

Lemma 1 (The Brunn-Minkowski inequality) [3] *If* $K, L \in \mathcal{K}^n$, *then*

$$V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{5}$$

with equality if and only if K *and* L *are homothetic.*

Lemma 2 (The Brunn-Minkowski inequality of the Quermassintegrals) [4, 5] *If* $K, L \in \mathcal{K}^n$, *and* $0 \leq i < n-1$, *then*

$$W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \tag{6}$$

with equality if and only if K *and* L *are homothetic.*

In 1993, Lutwak [6] established the Minkowksi inequality for mixed projection bodies as follows

The Minkowksi inequality for mixed projection bodies. If $K, L \in \mathcal{K}^n$, then

$$V(\Pi_1(K, L)) \geq V(\Pi K)^{(n-2)/(n-1)} V(\Pi L)^{1/(n-1)}, \tag{7}$$

with equality if and only if K and L are homothetic.

This is the special case $i = 0$ of:

The Minkowksi inequality of the Quermassintegrals for mixed projection bodies. If $K, L \in \mathcal{K}^n$, and $0 \leq i < n$, then

$$W_i(\Pi_1(K, L)) \geq W_i(\Pi K)^{(n-2)/(n-1)} W_i(\Pi L)^{1/(n-1)}, \tag{8}$$

with equality if and only if K and L are homothetic.

From (4), we obtain that

$$\Pi(K+\varepsilon L)=\sum_{i=0}^{n-1}\varepsilon^i\Pi_i(K,L)\binom{n-1}{i}$$

Hence, from Lemma 1

$$\begin{aligned}
&V(\Pi(K+\varepsilon L))^{1/n}\\
&=V\left(\Pi K+(n-1)\varepsilon\Pi_1(K,L)+\sum_{i=2}^{n-1}\varepsilon^i\Pi_i(K,L)\binom{n-1}{i}\right)^{1/n}\\
&\geq V(\Pi K+(n-1)\varepsilon\Pi_1(K,L))^{1/n}+V\left(\sum_{i=2}^{n-1}\varepsilon^i\Pi_i(K,L)\binom{n-1}{i}\right)^{1/n}\\
&\geq V(\Pi K)^{1/n}+(n-1)\varepsilon V(\Pi_1(K,L))^{1/n}+V\left(\sum_{i=2}^{n-1}\varepsilon^i\Pi_i(K,L)\binom{n-1}{i}\right)^{1/n}
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{V(\Pi(K+\varepsilon L))^{1/n}-V(\Pi K)^{1/n}}{\varepsilon}\\
&\geq (n-1)V(\Pi_1(K,L))^{1/n}+\lim_{\varepsilon\to 0}V\left(\sum_{i=2}^{n-1}\varepsilon^{i-1}\Pi_i(K,L)\binom{n-1}{i}\right)^{1/n}\\
&=(n-1)V(\Pi_1(K,L))^{1/n}, \qquad (9)
\end{aligned}$$

with equality if and only if K and L are homothetic.

Then, (7) and (9) will yield a new strengthened form of Minkowski inequality for mixed projection bodies as follows

Theorem 1. *If $K,L\in\mathcal{K}^n$, then*

$$V(\Pi K)^{(n-2)/(n-1)}V(\Pi L)^{1/(n-1)}\leq$$

$$V(\Pi_1(K,L))\leq\left(\frac{1}{n-1}\lim_{\varepsilon\to 0}\frac{V(\Pi(K+\varepsilon L))^{1/n}-V(\Pi K)^{1/n}}{\varepsilon}\right)^n,$$

where with equality if and only if K and L are homothetic.

Similarly, from (4) and (6), we also prove the following result.

Theorem 2 *If $K,L\in\mathcal{K}^n$, and $0\leq i<n$, then*

$$W_i(\Pi K)^{(n-2)/(n-1)}W_i(\Pi L)^{1/(n-1)}\leq$$

$$W_i(\Pi_1(K,L))\leq\left(\frac{1}{n-1}\lim_{\varepsilon\to 0}\frac{W_i(\Pi(K+\varepsilon L))^{1/(n-i)}-W_i(\Pi K)^{1/(n-i)}}{\varepsilon}\right)^{n-i},$$

where with equality if and only if K and L are homothetic.

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