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Some Function Spaces on $\mathbb{R}$

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This paper is devoted to the introduction and investigation of some function spaces. The elements are functions defined on $\mathbb{R}$, taking values in a Banach space.

These spaces are general enough to contain the L^p-spaces, as well as Marcinkiewicz's spaces $\mathcal{M}^p$. The last class of function spaces to be mentioned is that of almost periodic functions in the sense of Besicovitch. These spaces are closed subspaces of Marcinkiewicz' spaces.

Let E be a real Banach space, and consider those maps $x \in L^p_{loc}(\mathbb{R}, E)$, such that

$$\sup_{\ell > 0} \left\{ \rho(\ell) \int_{-\ell}^{\ell} |x(s)|^p ds \right\}^{1/p} < +\infty, \tag{0}$$

where $p \geq 1$, and $\rho(\ell)$ is a positive function defined on $\mathbb{R}_+ - \{0\}$. The norm appearing under the integral is the norm of E, while the integral in (0) stands for the Lebesgue-Bochner integral.

It can be easily shown that the left hand side of (0) defines a norm on a Banach space, we shall denote by $E^p_\rho(\mathbb{R}, E)$, which consists of all the maps $\mathbb{R} \longrightarrow E$, such that (0) is verified. The norm in $E^p_\rho(\mathbb{R}, E)$ will be denoted by $|\cdot|_{\rho,p}$ and appe ars in the left hand side of (0).

A general remark is that convergence in $E^p_\rho(\mathbb{R}, E)$ implies convergence in the underlying space $L_{loc}(\mathbb{R}, E)$.

Norms like $|\cdot|_{\rho,p}$, in case of spaces whose functions are defined on $\mathbb{R}_+$, have been recently considered, in view of applications, by M. Kwapisz [5]. It has been shown in [5] that the set $E^p_\rho(\mathbb{R}_+, E)$ is indeed a Banach space. The construction of these spaces has been motivated in [5] by the fact that the use of fixed point methods in the theory of functional equations can lead to global existence of solutions.

Before sketching the proof of the fact that $E^p_\rho(\mathbb{R}, E)$ is a Banach space ($p \geq 1$), let us point out that the classical Lebesgue spaces $L^p(\mathbb{R}, E)$, are obtained in case we choose $\rho(\ell) \equiv 1$.

Also, choosing $\rho(\ell) = (2\ell)^{-1}$, one obtains functions spaces richer than the Marcinkiewicz' space $\mathcal{M}^p(\mathbb{R}, E)$. For details concerning the space $\mathcal{M}^p(\mathbb{R}, E)$, $p \geq 1$, the reader can consult the book by Corduneanu [4], where the case $p = 2$ is discusse d.

The spaces of almost periodic functions in the sense of Besicovitch [4] are obtained as subspaces of $\mathcal{M}^p(\mathbb{R}, E)$. Namely, the space $B^p(\mathbb{R}, E)$, $p \geq 1$, is the closure in $\mathcal{M}^p(\mathbb{R}, E)$

of the linear manifold of trigonometric polynomials of the form

$$\mathbb{T}(t) = a_1 e^{i\lambda_1 t} + a_2 e^{i\lambda_2 t} + \cdots + a_n e^{i\lambda_n t}, \tag{1}$$

with $a_k \in E,\ k = 1, 2, ..., n$, and $\lambda_k \in \mathbb{R},\ k = 1, 2, ..., n$.

Other function spaces can be obtained by choosing the "weight function" in different ways. Such spaces distinguish themselves by a certain kind of behaviour at infinity for their functions.

In particular, the oscillatory character of these functions is assured for the functions belonging to the Besicovitch spces $B^p(\mathbb{R}, E),\ p \geq 1$. The functions belonging to the B^p-spaces have the remarkable property of possessing a "mean value", i.e., the existence of the limit

$$\lim_{\ell \longrightarrow \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} x(s)ds \in E$$

is always assured. As it is known [1], this fact leads to the construction of a (generalized) Fourier analysis for the Besicovitch almost periodic functions.

Proposition 1. *The set* $E^p_\rho(\mathbb{R}, E),\ p \geq 1$, *is a Banach space.*

Proof. We need to prove, first, that the map $x \longrightarrow |x|_{\rho,p}$ from E to $\mathbb{R}_+$, with $|x|_{\rho,p}$ given by the left hand side in (0), satisfies the following conditions:

a) $|x + y|_{\rho,p} \leq |x|_{\rho,p} + |y|_{\rho,p}$, for any $x, y \in E^p_\rho(\mathbb{R}, E)$;

b) $|x|_{\rho,p} \geq 0$, for any $x \in E^p_\rho(\mathbb{R}, E)$, the equal sign being valid only for $x = \theta \in E$, a.e. on $\mathbb{R}$, with θ the null element of the space E;

c) $|\lambda x|_{\rho,p} = |\lambda|\,|x|_{\rho,p}$, for any $\lambda \in \mathbb{R}$, and $x \in E^p_\rho(\mathbb{R}, E)$.

The properties are easy consequences of the corresponding ones for the norms in the space E and $L^p(\mathbb{R}, E)$. See also Kwapisz [5].

The only property of the norm $|x|_{\rho,p}$ we have to consider is the *completeness* of the linear normed space $E^p_\rho(\mathbb{R}, E)$ with respect to this norm.

Let $\{x_k;\ k \geq 1\} \subset E^p_\rho(\mathbb{R}, E)$ be a Cauchy sequence. This means that to each $\varepsilon > 0$, one can associate an integer $N = N(\varepsilon) > 0$, such that

$$|x_{k+j} - x_k|^p_\rho < \varepsilon \text{ for } k \geq N,\ j \geq 1. \tag{2}$$

We need to prove that there exists an element $x \in E^p_\rho(\mathbb{R}, E)$, such that

$$\lim_{k \longrightarrow \infty} x_k = x \text{ in } E^p_\rho(\mathbb{R}, E). \tag{3}$$

First, we notice that (2) is equivalent to the inequality

$$\sup \left\{ \rho(\ell) \int_{-\ell}^{\ell} |x_{k+j}(s) - x_k(s)|^p ds \right\}^{1/p} < \varepsilon, \tag{4}$$

for $k \geq N$ and $j \geq 1$. If one fixes now a number $\ell > 0$ (which can be arbitrarily large), then we derive frm (4)

$$\rho(\ell) \int_{-\ell}^{\ell} |x_{k+j}(s) - x_k(s)|^p ds < \varepsilon^p, \tag{5}$$

for $k \geq N,\ j \geq 1$. With $\ell > 0$ fixed, (5) is a Cauchy type condition for the restrictions to the interval $(-\ell, \ell)$ of the terms of the sequence $\{x_k;\ k \geq 1\}$. In other words, we have a Cauchy sequence on the inteval $(-\ell, \ell)$, in the space $L^p((-\ell, \ell), E)$. Consequently, since $L^p((-\ell, \ell), E)$ is a Banach space, we can infer the existence of an element $x_\ell \in L^p((-\ell, \ell), E)$, such that $x_k \longrightarrow x_\ell$, as $k \longrightarrow \infty$, in the space $L^p((-\ell, \ell), E)$. It is worth noticing that for $\ell_1 > \ell$, one has $x_{\ell_1} = x_\ell$ a.e. on $(-\ell, \ell)$, due to the fact $L^p((-\ell, \ell), E)$ can be regarded as a (closed) subspace of the space $L^p((-\ell, \ell), E)$.

Hence, we can define an element $x \in L^p_{loc}(\mathbb{R}, E)$, by letting $x = x_\ell$ a.e. on each interval $(-\ell, \ell) \in \mathbb{R}$. We have to prove that $x \in E^p_\rho(\mathbb{R}, E)$. In order to achieve this fact, we take (5) into account, and for fixed $k \geq N$, we let $j \longrightarrow \infty$. The result is the inequality

$$\rho(\ell) \int_{-\ell}^{\ell} |x(s) - x_k(s)|^p ds \leq \varepsilon^p, \tag{6}$$

valid for each $k \geq N$. Actually, (6) proves that $x - x_k \in L^p((-\ell, \ell), E)$, and since $x = (x - x_k) + x_k$, we obtain $x \in L^p((-\ell, \ell), E)$.

If in (6) we take both sides at the power p^{-1}, we obtain after taking the supremum, with respect to $\ell > 0$, in the left hand side

$$|x - x_k|_{\rho,p} \leq \varepsilon, \quad \text{for } k \geq N(\varepsilon), \tag{7}$$

The inequality (7) proves the fact that $x_k \longrightarrow x$ as $k \longrightarrow \infty$, in $E^p_\rho(\mathbb{R}, E)$.

This ends the proof of Proposition 1.

Remark. Based on the equivalence of two norms on a linear normed space, one can easily see that two different functions $\rho(\ell)$ and $\bar{\rho}(\ell)$, both positive on $\mathbb{R}_+ - \{0\}$, will generate the same space $E^p_\rho(\mathbb{R}, E)$ if there exist two positive co nstants α and β, such that for $\ell > 0$

$$\alpha\rho(\ell) \leq \bar{\rho}(\ell) \leq \beta\rho(\ell). \tag{8}$$

Indeed, one sees that condition (0) holds true, if and only if

$$\sup_{\ell > 0} \left\{ \bar{\rho}(\ell) \int_{-\ell}^{\ell} |x(s)|^p ds \right\}^{1/p} < +\infty. \tag{9}$$

Let us return now to the space $E^p_\rho(\mathbb{R}, E)$, with $\rho = (2\ell)^{-1}$, $\ell > 0$. In this case, in formula (0), the quantity between the brackets $\{\cdot\}$ is the mean value of the function $|x(t)|^p$, $p \geq 1$.

If instead of the supremum, taken in (0), one takes the limit as $\ell \longrightarrow \infty$, we are lead to the space $\mathcal{M}^p(\mathbb{R}, E)$, which is known as Marcinkiewicz's space. It has been mainly investigated in the case $p = 2$, and used in connection with the so-called " energetic stability" [2], [4].

More precisely, the space $\mathcal{M}^p(\mathbb{R}, E)$ is the set of maps from $\mathbb{R}$ into E, such that

$$\lim_{\ell \longrightarrow \infty} \left\{ (2\ell)^{-1} \int_{-\ell}^{\ell} |x(s)|^p ds \right\}^{1/p} \tag{10}$$

exists (in $\mathbb{R}$). The norm in $\mathcal{M}^p(\mathbb{R}, E)$ is defined by the quantity given in (10).

It is obvious that in case the limit in (10) exists, one has

$$\sup_{\ell>0}\left\{(2\ell)^{-1}\int_{-\ell}^{\ell}|x(s)|^p ds\right\}^{1/p} < \infty. \tag{11}$$

The inequality (11) shows that the following inclusion takes place, for any $p \geq 1$:

$$\mathcal{M}^p(\mathbb{R}, E) \subset E^p_\rho(\mathbb{R}, E), \rho(\ell) = (2\ell)^{-1}. \tag{12}$$

Finally, the Besicovitch space $B^p(\mathbb{R}, E)$ is defined as the closure in $\mathcal{M}^p(\mathbb{R}, E)$, of the set of trigonometric polynomials of the form (1).

Remark. It is worth mentionning that the limit in (10) may be zero, even though $x \neq 0$ a.e. on $\mathbb{R}$. Therefore, $|\cdot|^p_\rho$ is only a semi-norm on $B^p(\mathbb{R}, E)$. Actually, this space of almost periodic functions is the factor space obtained from $E^p_\rho(\mathbb{R}, E)$, $\rho = (2\ell)^{-1}$, with respect of the subspace consisting of those functions for which the limit in (10) is zero.

References

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