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Prolongation Of Solutions Of The Darboux Problem For Third Order Hyperbolic Inclusions

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Abstract. In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form $u_{xyz} \in F(x, y, z, u)$. It is proved a theorem of prolongation for the solutions of the considered problem and also an existence theorem for a saturated solution.

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1 Introduction

In this paper we consider the Darboux Problem for a third hyperbolic inclusion of the form

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} \in F(x,y,z,u), \qquad (1.1)$$
$$(x,y,z) \in D = [0,a]\times[0,b]\times[0,c], \; u \in \Omega \subset \mathbb{R}^n,$$

with initial values

$$\begin{cases} u(x,y,0) = \varphi(x,y), & (x,y) \in D_1 = [0,a]\times[0,b], \\ u(0,y,z) = \psi(y,z), & (y,z) \in D_2 = [0,b]\times[0,c], \\ u(x,0,z) = \chi(x,z), & (x,z) \in D_3 = [0,a]\times[0,c], \end{cases} \qquad (1.2)$$

where φ, ψ, χ are absolutely continuous in Carathéodory's sense [2, §565-§570], $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ and they satisfy the conditions

$$\begin{cases} u(x,0,0) = \varphi(x,0) = \chi(x,0) = v^1(x), \; x \in [0,a], \\ u(0,y,0) = \varphi(0,y) = \psi(y,0) = v^2(y), \; y \in [0,b], \\ u(0,0,z) = \psi(0,z) = \chi(0,z) = v^3(z), \; z \in [0,c], \\ u(0,0,0) = v^1(0) = v^2(0) = v^3(0) = v^0, \end{cases} \qquad (1.3)$$

where $F : D\times\Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex and non-empty values and $\Omega \subset \mathbb{R}^n$ is an open subset.

Under suitable assumptions, in [17] we prove an existence theorem for a local solution of the Darboux Problem (1.1)+(1.2) and that the set of its solutions is compact in Banach space $C(D_0;\mathbb{R}^n)$, $D_0 = [0, x_0]\times[0, y_0]\times[0, z_0] \subseteq D$; moreover, as a function of the initial values this set defines an upper semi-continuous multifunction.

In this paper, a prolongation theorem for the local solutions of the Darboux Problem (1.1)+(1.2), and also an existence theorem for a saturated solution of the same problem, are proved.

This study has been suggested by [16] and it provides an extension of the results in that article.

2 Preliminaries

The definitions and Theorems 2.1-2.4 in this section are taken from [2]-[5], [8]-[15].

Definition 2.1. Let X and Y be two non-empty sets. A *multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup\limits_{x\in X} \Phi(x)$ is the *range* of Φ. $\Phi(X) = \{\cup\Phi(x) \mid x \in X\}$.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.

a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup\limits_{x\in A} \Phi(x)$;

b) If $B \subset Y$, the *counterimage* of B by Φ is

$$\Phi^-(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The *graph* of Φ, denoted graph Φ, is set

$$\operatorname{graph}\Phi = \{(x, y) \in X\times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. Let us now take $\Phi : X \to 2^Y$. An element $x \in X$ with the property $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.4. A univalued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.5. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper semi-continuous* if, for any closed $B \subset Y$, $\Phi^-(B)$ is closed in X.

Definition 2.6. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* if $\Phi^-(B) \in \mathcal{F}$ for every closed subset $B \subset Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.

Theorem 2.1. [14] *Let X and Y be two metric spaces, Y compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

(*i*) *the multifunction Φ is upper semi-continuous;*

(ii) the graph of Φ *is a closed subset of* $X \times Y$*;*

(iii) any would be the sequences $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$*, from* $x_n \to x$*,* $y_n \to \Phi(x_n)$ *and* $y_n \to y$*, it follows that* $y \in \Phi(x)$*.*

Definition 2.7. [2], [8], [9], [15] The function $u : \Delta \to \mathbb{R}^n$, $\Delta \subset \mathbb{R}^2$ is *absolutely continuous in Carathéodory's sense* [2, §565-§570] if and only if $u(x, y)$ is continuous on Δ, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x, y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue-integrable on Δ.

Theorem 2.2. [2], [5], [15] *The function* $u : \Delta \to \mathbb{R}^n$, $\Delta = [0, a]\times[0, b] \subset \mathbb{R}^2$ *is absolutely continuous in Carathéodory's sense on* Δ *if and only if there exist* $f \in L^1(\Delta; \mathbb{R}^n)$*,* $g \in L^1([0, a]; \mathbb{R}^n)$*,* $h \in L^1([0, b]; \mathbb{R}^n)$ *such that*

$$u(x, y) = \int_0^x \int_0^y f(s, t)ds\, dt + \int_0^x g(s)ds + \int_0^y h(t)dt + u(0, 0).$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\Delta; \mathbb{R}^n)$ [8], [9]. In [5] this space is denoted by $AC(\Delta; \mathbb{R}^n)$.

Theorem 2.3. [5] *The space* $C^*(\Delta; \mathbb{R}^n)$ *endowed with the norm*

$$\|u(\cdot, \cdot)\| = \int_0^a \int_0^b \|u_{xy}(s, t)\|ds\, dt + \int_0^a \|u_x(s, 0)\|ds + \int_0^b \|u_y(0, t)\|dt + \|u(0, 0)\|,$$

where $\Delta = [0, a]\times[0, b]$ *and* $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

Definition 2.8. [2], [10] The function $u : D \subset \mathbb{R}^3 \to \mathbb{R}^n$ is *absolutely continuous in Carathéodory's sense* [2, §565-§570] if and only if $u(x, y, z)$ is continuous on D, absolutely continuous in each variable (for any pair of the other two variables), and similarly for $u_x(x, y, z), u_y(x, y.z), u_z(x, y, z)$, $u_{xy}(x, y, z), u_{yz}(x, y, z), u_{xz}(x, y, z)$ and u_{xyz} is Lebesgue-integrable.

Theorem 2.4. [5], [10] *The function* $u : D \to \mathbb{R}^n$, $D = [0, a]\times[0, b]\times[0, c] \subset \mathbb{R}^3$, *is absolutely continuous in Carathéodory's sense on* D *if and only if there exist* $f \in L^1(D; \mathbb{R}^n)$*,* $g_1 \in L^1(D_1; \mathbb{R}^n)$*,* $g_2 \in L^1(D_2; \mathbb{R}^n)$*,* $g_3 \in L^1(D_3; \mathbb{R}^n)$*,* $h_1 \in L^1([0, a]; \mathbb{R}^n)$*,* $h_2 \in L^1([0, b]; \mathbb{R}^n)$*;* $h_3 \in L^1([0, c]; \mathbb{R}^n)$*,* $D_1 \in [0, a]\times[0, b]$*,* $D_2 = [0, b]\times[0, c]$*,* $D_3 = [0, a]\times[0, c]$ *such that*

$$\begin{aligned} u(x, y, z) \quad &= \int_0^x \int_0^y \int_0^z f(r, s, t)dr\, ds\, dt + \int_0^x \int_0^y g_1(r, s)dr\, ds + \\ &+ \int_0^y \int_0^z g_2(s, t)ds\, dt + \int_0^x \int_0^z g_3(r, t)dr\, dt + \\ &+ \int_0^x h_1(r)dr + \int_0^y h_2(s)ds + \int_0^z h_3(t)dt + u(0, 0, 0). \end{aligned}$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(D; \mathbb{R}^n)$ [10].

Theorem 2.5. [5], [10] *The space $C^*(D;\mathbb{R}^n)$ endowed with the norm*

$$\begin{aligned}\|u(\cdot,\cdot,\cdot)\| &= \int_0^a\int_0^b\int_0^c \|u_{xyz}(r,s,t)\|dr\,ds\,dt + \int_0^a\int_0^b \|u_{xy}(r,s,0)\|dr\,ds+\\ &+\int_0^b\int_0^c \|u_{yz}(0,s,t)\|ds\,dt + \int_0^a\int_0^c \|u_{xz}(r,0,t)\|dr\,dt+\\ &+\int_0^a \|u_x(r,0,0)\|dr + \int_0^b \|u_y(0,s,0)\|ds+\\ &+\int_0^c \|u_z(0,0,t)\|dt + \|u(0,0,0)\|,\end{aligned}$$

where $\|\cdot\|$ is the Euclidean norm, is a Banach space.

3 Results

In [17] the notion of a *local solution* for the Darboux Problem (1.1)+(1.2) is defined and it is proved an existence theorem for a local solution of this problem, together with some properties of the set of its solutions, namely that this set is a compact subset in Banach space $C(D_0;\mathbb{R}^n)$ and, as a function of initial values, it defines an upper semi-continuous multifunction on $D_0 = [0,x_0]\times[0,y_0]\times[0,z_0] \subseteq D$.

Let the following hypotheses be satisfied:

(H_1) $F\times\Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex, non-empty values in $\mathbb{R}^n$, $D = [0,a]\times[0,b]\times[0,c] \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^n$ is an open subset;

(H_2) For any $(x,y,z) \in D$, the mapping $u \to F(x,y,z,u)$ is upper semi-continuous on Ω;

(H_3) For any $u \in \Omega$, the mapping $(x,y,z) \to F(x,y,z,u)$ is Lebesgue-measurable on D;

(H_4) There exists a function $k : D \to \mathbb{R}_+,\ k \in \mathcal{L}^1(D;\mathbb{R}_+)$ such that

$$\|\zeta\| \le k(x,y,z),\ (\forall)\zeta \in F(x,y,z,u),\ (\forall)(x,y,z) \in D,\ (\forall)u \in \Omega;$$

(H_5) The functions $\varphi \in C^*(D_1;\mathbb{R}^n),\ \psi \in C^*(D_2;\mathbb{R}^n),\ \chi \in C^*(D_3;\mathbb{R}^n)$ are absolutely continuous in Carathéodory's sense functions and satisfy conditions (1.3).

Remark 1. The function $\alpha : D \to \mathbb{R}^n$ defined by

$$\begin{aligned}\alpha(x,y,z) &= \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y)-\\ &- \psi(0,z) + \psi(0,0) = \varphi(x,y) + \psi(y,z) + \chi(x,z)-\\ &- v^1(x) - v^2(y) - v^3(z) + v^0,\end{aligned} \tag{3.1}$$

is an absolutely continuous in Carathéodory's sense function on D, $\alpha \in C^*(D;\mathbb{R}^n)$ [2, §565-§570].

Remark 2. Denote by $M \subset \Omega$ the convex compact set in which the function $\alpha : D \to \mathbb{R}^n$, defined by (3.1), takes its values for all $(x,y,z) \in D_0$.

Remark 3. Let $(x_0, y_0, z_0) \in]0, a]\times]0, b]\times]0, c]$ be a point such that

$$\int_0^{x_0}\int_0^{y_0}\int_0^{z_0} k(r,s,t)dr\,ds\,dt < d(M, C_\Omega),$$

where $d(M, C_\Omega)$ is the distance from M to $C_\Omega = \mathbb{R}^n - \Omega$, an inequality immediately resulting from the integrability of function k.

Definition 3.1. The Darboux Problem for the hyperbolic inclusion (1.1) means to determine a *solution* of this inclusion which satisfies the initial conditions (1.2).

Definition 3.2. A *local solution* of Darboux Problem (1.1)+(1.2) is defined as a function $U : D_0 \to \Omega$, $U \in C^*(D_0; \mathbb{R}^n)$, absolutely continuous in Carathéodory's sense [2, §565-§570], which satisfies (1.1) for a.e. $(x, y, z) \in D_0$, and also initial conditions (1.2) for all $(x, y) \in [0, x_0]\times[0, y_0]$, all $(y, z) \in [0, y_0]\times[0, z_0]$, all $(x, z) \in [0, x_0]\times[0, z_0]$.

In [17] it is proved the following:

Theorem 3.1. *Let the hypotheses* $(\mathrm{H}_1) - (\mathrm{H}_5)$ *be satisfied. Then:*

(i) *there exists at least a local solution U of Darboux Problem (1.1)+(1.2);*

(ii) *the set S_a of the local solutions U is compact in the Banach space $C(D_0; \mathbb{R}^n)$;*

(iii) *the multifunction $\alpha \to S_a$ is upper semi-continuous on $C^*(D_0; \mathbb{R}^n)$, taking values in $C(D_0; \mathbb{R}^n)$.*

The solution U is a fixed point of a suitable multifunction which satisfies the Kakutani-Ky Fan fixed point Theorem.

In this paper we prove theorems of prolongation for the solutions to the Darboux Problem (1.1)+(1.2) and of existence for a saturated solution.

Definition 3.3. A local solution for the Darboux Problem (1.1)+(1.2), $U : D_0 \to \Omega$ is *prolongable* (*non-saturated*) if there exists a solution $\widetilde{U} : \widetilde{D} \to \mathbb{R}^n$ for the Darboux Problem (1.1)+(1.2) such that

$$\begin{cases} D_0 \subseteq \widetilde{D},\ D_0 \neq \widetilde{D}, \\ \widetilde{U}(x, y, z) = U(x, y, z),\ (x, y, z) \in D_0, \end{cases}$$

where $\widetilde{D} \subseteq D$ is a union of D_0 with a finite number of adjacent parallelepipeds.

Theorem 3.2. *Let the hypotheses* $(\mathrm{H}_1) - (\mathrm{H}_5)$ *be satisfied together with the hypotheses:*

(H_6) *The set Ω is bounded, that is there exists a constant $C \in \mathbb{R}_+$ such that $\|u\| \leq C$, $(\forall) u \in \Omega$;*

(H_7) *The multifunction F maps bounded sets onto bounded sets, hence a constant $K \in \mathbb{R}_+$ exists such that*

$$\sup\{\|\zeta\| \mid \zeta \in F(x, y, z, u)\} \leq K$$

for any $(x, y, z, u) \in D\times\Omega$.

Then the local solution U is prolongable.

Proof. In view of Theorem 3.1 [17] at least a local solution $U : D_0 \to \Omega$ of the Darboux Problem (1.1)+(1.2) exists. We are going to show that the local solution U can be prolonged over a parallelepiped Δ_1 adjacent to D_0 on the surface of equation $x = x_0$. Let $(x_0, \bar{y}, \bar{z})$, $0 \le x_0 \le a$, $0 \le \bar{y} \le y_0 \le b$, $0 \le \bar{z} \le z_0 \le c$ a point on the boundary $\partial_1 D_0$ for D_0 of equation $x = x_0$. Using Cauchy's criterion for characterizing the functions with finite limits, we will deduce that solution $U : D_0 \to \Omega$ has a finite limit at $(x_0, \bar{y}, \bar{z})$. Indeed, let (x', y', z') and (x'', y'', z'') be two points of D_0 included in a neighbourhood of $(x_0, \bar{y}, \bar{z})$. In view of Theorem 3.1 [17], we have

$$\begin{aligned} U(x',y',z') &= \alpha(x',y',z') + \int_0^{x'}\int_0^{y'}\int_0^{z'} \beta(r,s,t)dr\,ds\,dt,\ (x',y',z') \in D_0 \\ U(x'',y'',z'') &= \alpha(x'',y'',z'') + \int_0^{x''}\int_0^{y''}\int_0^{z''} \beta(r,s,t)dr\,ds\,dt,\ (x'',y'',z'') \in D_0 \end{aligned} \tag{3.2}$$

where

$$\beta(x,y,z) \in \Gamma(x,y,z) \subseteq F(x,y,z,U(x,y,z)), \quad \text{for a.e. } (x,y,z) \in D_0, \tag{3.3}$$

β is a measurable selection of the multifunction $\Gamma : D \to \mathcal{C}\left(\mathbb{R}^n\right)$ [3], [4], [17].

We obtain by (3.2)

$$\begin{aligned} &\|U(x',y',z') - U(x'',y'',z'')\| \le \|\alpha(x',y',z') - \alpha(x'',y'',z'')\| + \\ &+ \left\| \int_0^{x'}\int_0^{y'}\int_0^{z'} \beta(r,s,t)dr\,ds\,dt - \int_0^{x''}\int_0^{y''}\int_0^{z''} \beta(r,s,t)dr\,ds\,dt \right\| = \\ &= \|\alpha(x',y',z') - \alpha(x'',y'',z'')\| + \\ &+ \left\| \int_0^{x'}\int_0^{y'}\int_{z'}^{z''} \beta(r,s,t)dr\,ds\,dt + \int_{x'}^{x''}\int_0^{y'}\int_0^{z''} \beta(r,s,t)dr\,ds\,dt + \right. \\ &\left. + \int_{x'}^{x''}\int_{y'}^{y''}\int_0^{z''} \beta(r,s,t)dr\,ds\,dt + \int_0^{x'}\int_{y'}^{y''}\int_0^{z''} \beta(r,s,t)dr\,ds\,dt \right\| \le \\ &\le \|\alpha(x',y',z') - \alpha(x'',y'',z'')\| + \\ &+ \left| \int_0^{x'}\int_0^{y'}\int_{z'}^{z''} \|\beta(r,s,t)\|dr\,ds\,dt \right| + \left| \int_{x'}^{x''}\int_0^{y'}\int_0^{z''} \|\beta(r,s,t)\|dr\,ds\,dt \right| + \\ &+ \left| \int_{x'}^{x''}\int_{y'}^{y''}\int_0^{z''} \|\beta(r,s,t)\|dr\,ds\,dt \right| + \left| \int_0^{x'}\int_{y'}^{y''}\int_0^{z''} \|\beta(r,s,t)\|dr\,ds\,dt \right|. \end{aligned} \tag{3.4}$$

But the function α is absolutely continuous on D, hence it is continuous on D [2, §565-§570] and continuous on $D_0 \subseteq D$. It follows that $(\forall)\varepsilon > 0$, $(\exists)\delta = \delta(\varepsilon) > 0$ such that

$$\|(x',y',z') - (x'',y'',z'')\| < \delta(\varepsilon) \Longrightarrow \|\alpha(x',y',z') - \alpha(x'',y'',z'')\| < \varepsilon. \tag{3.5}$$

It follows from hypothesis (H_7) and (3.3) that

$$\|\beta(r,s,t)\| \le K, \text{ for a.e. } (r,s,t,) \text{ in } D_0. \tag{3.6}$$

From (3.4), (3.5), (3.6) we obtain

$$\begin{aligned}&\|U(x',y',z')-U(x'',y'',z'')\|\le\\&\quad\le\varepsilon+Kx'y'|z'-z''|+Ky'z''|x'-x''|+K|x'-x''||y'-y''|z''+\\&\quad+Kx'z''|y'-y''|<\varepsilon+Kab\delta(\varepsilon)+Kbc\delta(\varepsilon)+Kc\delta^2(\varepsilon)+\\&\quad+Kac\delta(\varepsilon)=\varepsilon+K\delta(\varepsilon)[ab+bc+c\delta(\varepsilon)+ac]=\bar{\varepsilon},\end{aligned}\tag{3.7}$$

for $(x',y',z'),(x'',y'',z'')\in D_0$ and $\|(x',y',z')-(x'',y'',z'')\|<\delta(\varepsilon)$.

Therefore the Cauchy criterion shows that there exists

$$\lim_{(x,y,z)\to(x_0,\bar{y},\bar{z})}U(x,y,z)=\overline{\psi}.\tag{3.8}$$

Taking into account the equality (3.8) we can define a function $\overline{\psi}$ on the boundary $\partial_1 D_0$ of equation $x=x_0$ of the parallelepiped D_0, that is absolutely continuous with respect to $(\bar{y},\bar{z})\in[0,y_0]\times[0,z_0]$.

Using Theorem 3.1 [17], the local solution U can be written under the form

$$\begin{aligned}U(x,y,z)&=\alpha(x,y,z)+\int_0^x\int_0^y\int_0^z\beta(r,s,t)dr\,ds\,dt=\\&=\varphi(x,y)+\psi(y,z)+\chi(x,z)-\varphi(x,0)-\varphi(0,y)-\\&-\psi(0,z)+\psi(0,0)+\int_0^x\int_0^y\int_0^z\beta(r,s,t)dr\,ds\,dt.\end{aligned}\tag{3.9}$$

But from hypothesis (H_5) and Theorem 2.2, φ,ψ,χ can be written under the forms

$$\begin{aligned}\varphi(x,y)&=\int_0^x\int_0^y\varphi_{xy}(r,s)dr\,ds+\int_0^x\varphi_x(r,0)dr+\\&+\int_0^y\varphi_y(0,s)ds+\varphi(0,0),\ (x,y)\in D_1,\end{aligned}\tag{3.10}$$

$$\begin{aligned}\psi(y,z)&=\int_0^y\int_0^z\psi_{yz}(s,t)ds\,dt+\int_0^y\psi_y(s,0)ds+\\&+\int_0^z\psi_z(0,t)dt+\psi(0,0),\ (y,z)\in D_2,\end{aligned}\tag{3.11}$$

$$\begin{aligned}\chi(x,z)&=\int_0^x\int_0^z\chi_{xz}(r,t)dr\,dt+\int_0^x\chi_x(r,0)dr+\\&+\int_0^z\chi_z(0,t)dt+\chi(0,0),\ (x,z)\in D_3.\end{aligned}\tag{3.12}$$

We obtain

$$\varphi(0,y)=\int_0^y\varphi_y(0,s)ds+\varphi(0,0),\ \varphi(x,0)=\int_0^x\varphi_x(r,0)dr+\varphi(0,0),\tag{3.10$'$}$$

$$\psi(0,z)=\int_0^z\psi_z(0,t)dt+\psi(0,0),\ \psi(y,0)=\int_0^y\psi_y(s,0)ds+\psi(0,0).\tag{3.11$'$}$$

Replacing (3.10), (3.11), (3.12), (3.10′), (3.11′) in (3.9) we get

$$\begin{aligned} U(x,y,z) &= \int_0^x\int_0^y\int_0^z \beta(r,s,t)dr\,ds\,dt + \int_0^x\int_0^y \varphi_{xy}(r,s)dr\,ds+ \\ &+\int_0^y\int_0^z \psi_{yz}(s,t)ds\,dt + \int_0^x\int_0^z \chi_{xz}(r,t)dr\,dt+ \\ &+\int_0^y \psi_y(s,0)ds + \int_0^x \chi_x(r,0)dr + \int_0^z \chi_z(0,t)dt- \\ &-\varphi(0,0)+\psi(0,0)+\chi(0,0) = \\ &= \int_0^x\int_0^y\int_0^z \beta(r,s,t)dr\,ds\,dt + \int_0^x\int_0^y \varphi_{xy}(r,s)dr\,ds+ \\ &+\int_0^y\int_0^z \psi_{yz}(s,t)ds\,dt + \int_0^x\int_0^z \chi_{xz}(r,t)dr\,dt + \int_0^x \varphi_x(r,0)dr+ \\ &+\int_0^y \psi_y(s,0)ds + \int_0^z \chi_z(0,t)dt - \varphi(0,0)+\psi(0,0)+\chi(0,0). \end{aligned} \tag{3.13}$$

In order to obtain the expression of the function $\overline{\psi}(x_0,\bar{y},\bar{z})$, we have passed to the limit of (3.8) for $x \to x_0$, $y \to \bar{y}$, $z \to \bar{z}$ in equation (3.13). We take into account the absolute continuity of the integrals. We obtain similarly with [16]:

$$\begin{aligned} &\lim_{(x,y,z)\to(x_0,\bar{y},\bar{z})}\int_0^x\int_0^y\int_0^z \beta(r,s,t)dr\,ds\,dt = \int_0^{x_0}\int_0^{\bar{y}}\int_0^{\bar{z}} \beta(r,s,t)dr\,ds\,dt, \\ &\lim_{(x,y)\to(x_0,\bar{y})}\int_0^x\int_0^y \varphi_{xy}(r,s)dr\,ds = \int_0^{x_0}\int_0^{\bar{y}} \varphi_{xy}(r,s)dr\,ds, \\ &\lim_{(y,z)\to(\bar{y},\bar{z})}\int_0^y\int_0^z \psi_{yz}(s,t)ds\,dt = \int_0^{\bar{y}}\int_0^{\bar{z}} \psi_{yz}(s,t)ds\,dt, \\ &\lim_{(x,z)\to(x_0,\bar{z})}\int_0^x\int_0^z \chi_{xz}(r,t)dr\,dt = \int_0^{x_0}\int_0^{\bar{z}} \chi_{xz}(r,t)dr\,dt, \\ &\lim_{x\to x_0}\int_0^x \varphi_x(r,0)dr = \int_0^{x_0} \varphi_x(r,0)dr, \\ &\lim_{y\to\bar{y}}\int_0^y \psi_y(s,0)ds = \int_0^{\bar{y}} \psi_y(s,0)ds, \\ &\lim_{z\to\bar{z}}\int_0^z \chi_z(0,t)dt = \int_0^{\bar{z}} \chi_z(0,t)dt. \end{aligned} \tag{3.14}$$

From (3.8), (3.13), (3.14) we have

$$\begin{aligned}\overline{\psi}(x_0,\bar{y},\bar{z}) &= \lim_{(x,y,z)\to(x_0,\bar{y},\bar{z})} U(x,y,z) = \int_0^{x_0}\int_0^{\bar{y}}\int_0^{\bar{z}} \beta(r,s,t)dr\,ds\,dt+ \\ &+\int_0^{x_0}\int_0^{\bar{y}} \varphi_{xy}(r,s)dr\,ds + \int_0^{\bar{y}}\int_0^{\bar{z}} \psi_{yz}(s,t)ds\,dt+ \\ &+\int_0^{x_0}\int_0^{\bar{z}} \chi_{xz}(r,t)dr\,dt + \int_0^{x_0} \varphi_x(r,0)dr + \int_0^{\bar{y}} \psi_y(s,0)ds+ \\ &+\int_0^{\bar{z}} \chi_z(0,t)dt - \varphi(0,0)+\psi(0,0)+\chi(0,0),\ \ (x_0,\bar{y},\bar{z})\in\partial_1 D_0.\end{aligned} \tag{3.15}$$

Hence, from Theorem 2.4, the function $\overline{\psi}$ is abolutely continuous in Carathéodory's sense with respect to $(\bar{y},\bar{z})\in[0,y_0]\times[0,z_0]$.

We proceed similarly to show that the local solution U of the Darboux Problem (1.1)+(1.2) has a finite limit on the boundary $\partial_2 D_0$ of equation $y=y_0$ of D_0 and also on the boundary $\partial_3 D_0$ of equation $z=z_0$ of D_0.

Thus we obtain three functions $\overline{\varphi}(\bar{x},\bar{y},z_0)$, $\overline{\psi}(x_0,\bar{\bar{y}},\bar{\bar{z}})$, $\chi(\bar{\bar{\bar{x}}},y_0,\bar{\bar{\bar{z}}})$ that are absolutely continuous in Carathéodory's sense [2, §565–§570], $\overline{\varphi}\in C^*(D_1;\mathbb{R}^n)$, $\overline{\psi}\in C^*(D_2;\mathbb{R}^n)$, $\overline{\chi}\in C^*(D_3;\mathbb{R}^n)$ and respectively defined on the boundaries of equations $z=z_0$, $x=x_0$ and $y=y_0$ of D_0.

The local solution $U: D_0\to\mathbb{R}^n$ can be prolonged on a parallelepiped Δ_1 adjacent to D_0 on the boundary $\partial_3 D_0$ of equation $z=z_0$ in the following way. If $z_0<c$, we use Theorem 3.1 [17] of existence for a local solution of the Darboux Problem (1.1)+(1.2) in the parallelepiped

$$\Delta_1' = \{(x,y,z)\mid 0\le x\le x_0,\ 0\le y\le y_0,\ z_0\le z\le c\}$$

with the initial conditions

$$\begin{cases} U(\bar{x},\bar{y},z_0)=\overline{\varphi}(\bar{x},\bar{y},z_0), & (\bar{x},\bar{y})\in[0,x_0]\times[0,y_0],\\ U(0,y,z)=\psi(y,z), & (y,z)\in[0,y_0]\times[z_0,c],\\ U(x,0,z)=\chi(x,z), & (x,z)\in[0,x_0]\times[z_0,c].\end{cases}$$

Thus we obtain a local solution of the Darboux Problem (1.1)+(1.2) in a parallelepiped $\Delta_1\subseteq\Delta_1'$, which is a prolongation of the solution U.

Similarly, a local solution $U: D_0\subset\mathbb{R}^3\to\mathbb{R}^n$ can be prolonged on a parallelepiped Δ_2 adjacent to D_0 on the boundary $\partial_1 D_0$ of equation $x=x_0$. If $x_0<a$, we use Theorem 3.1 [17] of existence of local solution to the Darboux Problem (1.1)+(1.2) in the parallelepiped

$$\Delta_2' = \{(x,y,z)\mid x_0\le x\le a,\ 0\le y\le y_0,\ 0\le z\le z_0\}$$

with the initial conditions

$$\begin{cases} U(x_0,\bar{\bar{y}},\bar{\bar{z}})=\overline{\psi}(x_0,\bar{\bar{y}},\bar{\bar{z}}), & (\bar{\bar{y}},\bar{\bar{z}})\in[0,y_0]\times[0,z_0],\\ U(0,y,z)=\psi(y,z), & (y,z)\in[0,y_0]\times[0,z_0],\\ U(x,0,z)=\chi(x,z), & (x,z)\in[x_0,a]\times[0,z_0].\end{cases}$$

Thus we obtain a local solution of the Darboux Problem (1.1)+(1.2) in a parallelepiped $\Delta_2 \subseteq \Delta_2'$, which is a prolongation of the solution U.

Similarly, a local solution U to the Darboux Problem (1.1)+(1.2) can be prolonged on a parallelepiped Δ_3 adjacent to D_0 on the boundary $\partial_2 D_0$ of equation $y = y_0$. If $y_0 < b$ we apply Theorem 3.1 [17] of existence for a local solution of the Darboux Problem (1.1)+(1.2) in the parallelepiped

$$\Delta_3' = \{(x, y, z) \mid 0 \le x \le x_0,\ y_0 \le y \le b,\ 0 \le z \le z_0\}$$

with the initial conditions

$$\begin{cases} U(\overline{\overline{\overline{x}}}, y_0, \overline{\overline{\overline{z}}}) = \overline{\chi}(\overline{\overline{\overline{x}}}, \overline{\overline{\overline{z}}}), & (\overline{\overline{\overline{x}}}, \overline{\overline{\overline{z}}}) \in [0, x_0]\times[0, z_0], \\ U(0, y, z) = \varphi(y, z), & (y, z) \in [y_0, b]\times[0, z_0], \\ U(x, y, 0) = \varphi(x, y), & (x, y) \in [0, x_0]\times[y_0, b]. \end{cases}$$

It is obtained a local solution of the Darboux Problem (1.1)+(1.2) in a parallelepiped $\Delta_3 \subseteq \Delta_3'$, which is a prolongation of the solution U.

It follows that the local solution U can be prolonged on a domain Δ which is a union of D_0 with a finite number of adjacent parallelepipeds.

Theorem 3.3. *We assume the hypotheses* (H_1)–(H_7) *of Theorem* 3.2 *to be satisfied. If* $U : D_0 \to \Omega$ *is a local solution of the Darboux Problem (1.1)+(1.2) that is non-saturated* (*hence prolongable*), *then there exists a saturated solution* $U^* : D^* \to \Omega$ *of the Darboux Problem (1.1)+(1.2) such that*

$$\begin{cases} D_0 \subseteq D^*,\ D_0 \ne D^*,\ D^* \subseteq D \\ U^*(x, y, z) = U(x, y, z),\ (x, y, z) \in D_0, \end{cases}$$

hence U^* *is a prolongation of* U *onto* D^* *that has been built from* D_0 *joined with a union of parallelepipeds adjacent to* D_0.

Proof. Since, by hypothesis, solution U is non-saturated (prolongable), it follows that at least a solution $U' : D' \to \Omega$ of the Darboux Problem (1.1)+(1.2) exists, this U' being a prolongation of U on $D' \supseteq D_0$ in view of Theorem 3.2

$$\begin{cases} D_0 \subseteq D',\ D_0 \ne D',\ D' \subseteq D \\ U'(x, y, z) = U(x, y, z),\ (x, y, z) \in D_0, \end{cases}$$

where D' consists of D_0 plus a union of parallelepipeds that are adjacent to D_0.

We denote by S the set of the solutions $U_i : D_i' \to \Omega$ of the Darboux Problem (1.1)+(1.2), defined on the family of sets $\{D_i'\}_{i\in I}$, where I is a family of indices and each D_i' consists of D_0 plus a union of parallelepipeds adjacent to D_0, $D_i' \subseteq D$. The solutions U_i, $i \in I$, are prolongations of the solution U onto D_i', $i \in I$, that can be built by the method used in the proof of Theorem 3.2; hence we get the relations

$$\begin{cases} D_0 \subseteq D_i',\ D_0 \ne D_i',\ D_i' \subseteq D, \\ U_i(x, y, z) = U(x, y, z),\ (x, y, z) \in D_0, \quad \text{for } (\forall) i \in I. \end{cases}$$

The set S is non-empty since there exists $U' \in S$. We define, on the set S, a partial ordering relation in the following way. For two arbitrary solutions $U_i : D'_i \to \Omega$, $U_k : D'_k \to \Omega$, $U_i, U_k \in S$, we denote $U_i \prec U_k$, $i, k \in I$, if we have

$$\begin{cases} D'_i \subseteq D'_k, \\ U_i(x,y,z) = U_k(x,y,z), \ (x,y,z) \in D'_i. \end{cases}$$

The existence of a saturated solution of the Darboux Problem (1.1)+(1.2) is equivalent with the existence of a maximal element of the partially ordered set S.

We select an arbitrary chain $L \subset S$, hence a family $\{U_j\}_{j\in J}$, $J \subseteq I$, of solutions to the Darboux Problem (1.1)+(1.2), $U_j : D'_j \to \Omega$, $D_0 \subseteq D'_j$, $D_0 \neq D'_j$, $(\forall) j \in J$, which is totally ordered by the partial ordering earlier introduced. It follows that $\{D'_j\}_{j\in J}$ is a chain in the set $\pi(D)$ of the subsets of D that is partial ordered by the containment relation. Since $\pi(D)$ is an inductively ordered set, the chain $\{D'_j\}_{j\in J}$ admits the upper-bounding element $G = \bigcup\limits_{j\in J} D'_j$. We have $G \subseteq D$, $D_0 \subseteq G$, $D_0 \neq G$.

We determine an upper-bounding element for the chain L, hence a function $U_{\text{maj}} : G \to \Omega$, which is a solution to the Darboux Problem (1.1)+(1.2) on the set $G \subseteq D$ such that, for any $j \in J$, $U_j \prec U_{\text{maj}}$. Let $(x,y,z) \in G$. Then there exists at least a $j_0 \in J$ such that $(x,y,z) \in D'_{j_0}$. We define

$$U_{\text{maj}}(x,y,z) = U_{j_0}(x,y,z), \ (x,y,z) \in G.$$

But $U_{j_0} : D'_{j_0} \to \Omega$ is a solution of the Darboux Problem (1.1)+(1.2). It follows that U_{maj} has the same property, hence U_{maj} is a solution of the Darboux Problem (1.1)+(1.2) on G and

$$\begin{cases} D'_j \subseteq G, \ D'_j \neq G, \ (\forall) j \in J, \\ U_{\text{maj}}(x,y,z) = U_j(x,y,z), \quad \text{for } (x,y,z) \in D'_j, \end{cases}$$

then, by the partial ordering relation we have defined, $U_j \prec U_{\text{maj}}$, $(\forall) j \in J$. It follows that the chain $L \subseteq S$ is upper-bounded, whence it follows, in view of Zorn's theorem that S contains a maximal element which is the saturated solution $U^* : D^* \to \Omega$, where D^* consists of D_0 plus a union of parallelepipeds adjacent to D_0, $D_0 \subseteq D^*$, $D_0 \neq D^*$, $D^* \subseteq D$.

Theorem 3.4. *Let the hypotheses* (H_1)-(H_7) *of Theorem* 3.2 *be satisfied. If the saturated solution* U^* *is bounded on* D^**, then* $D^* = D$.

Proof. We assume by reductio ad absurdum that $D^* \subseteq D$, $D^* \neq D$. Then, by the procedure of prolongation in Theorem 3.2, we can prolong U^* on $D_1^* \supseteq D^*$, $D_1^* \neq D^*$, which is contradictory with that U^* is saturated. Hence $D^* = D$.

Theorem 3.5. *Let the hypotheses* (H_1)-(H_7) *of Theorem* 3.2 *be satisfied together with the hypothesis*

(H_8) *The multifunction* $F : D \times \Omega \to 2^{\mathbb{R}^n}$ *is sub-linear, hence two constants* $k_1 > 0$ *and* $k_2 \in \mathbb{R}$ *exist with the property*

$$\sup\{\|\zeta\| \mid \zeta \in F(x,y,z,u)\} \leq k_1\|u\| + k_2, \quad \textit{for a.e. } (x,y,z) \in D, \ u \in \Omega. \tag{3.16}$$

Then the saturated solution $U^* : D \to \Omega$ is bounded on D.

Proof. The saturated solution $U^* : D \to \Omega$ has by theorem 3.1 [17] the form

$$U^*(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \beta^*(r,s,t) dr\, ds\, dt,\ (x,y,z) \in D, \tag{3.17}$$

where $\alpha(x,y,z)$ is given by (3.1) and β^* is a measurable selection of the multivalued mapping Γ^* [3], [4] defined on D with compact non-empty values in $\mathbb{R}^n$ such that

$$\beta^*(x,y,z) \in \Gamma^*(x,y,z) \subseteq F(x,y,z,U^*(x,y,z)) \text{ for a.e. } (x,y,z) \in D. \tag{3.18}$$

From hypothesis (H_8) it follows that we have an inequality of the form (3.16) for the selection β^* since (3.16) holds; hence

$$\sup \|\beta^*(x,y,z)\| \le k_1 \|U^*(x,y,z)\| + k_2,\ \text{for a.e. } (x,y,z) \in D. \tag{3.19}$$

We obtain from (3.17) and (3.19) the inequality

$$\begin{aligned} \|U^*(x,y,z)\| &\le \sup \|\alpha(x,y,z)\| + \\ &+ k_1 \int_0^x \int_0^y \int_0^z \|U^*(r,s,t)\| dr\, ds\, dt + |k_2| xyz \le \\ &\le \sup \|\alpha(x,y,z)\| + |k_2| abc + k_1 \int_0^x \int_0^y \int_0^z \|U^*(r,s,t)\| dr\, ds\, dt, \end{aligned} \tag{3.20}$$

for a.e. $(x,y,z) \in D$.

Then, using the notation

$$B = \sup \|\alpha(x,y,z)\| + |k_2| abc,\quad (x,y,z) \in D,$$

the inequality (3.20) becomes

$$\|U^*(x,y,z)\| \le B + k_1 \int_0^x \int_0^y \int_0^z \|U^*(r,s,t)\| dr\, ds\, dt,\quad \text{for a.e. } (x,y,z) \in D. \tag{3.21}$$

It is possible to apply a Gronwall-type inequality [1], [6], [7] what leads to

$$\begin{aligned} \|U^*(x,y,z)\| &\le \\ &\le B \left[1 + k_1 \int_0^x \int_0^y \int_0^z \exp\left(\int_r^x \int_s^y \int_t^z k_1 d\xi\, d\eta\, d\zeta \right) dr\, ds\, dt \right] = \\ &= B \left[1 + k_1 \int_0^x \int_0^y \int_0^z \exp(k_1 (x-r)(y-s)(z-t)) dr\, ds\, dt \right] \le \\ &\le B \left[1 + k_1 \int_0^x \int_0^y \int_0^z \exp(k_1 xyz) dr\, ds\, dt \right] \le \\ &\le B \left[1 + k_1 \exp(k_1 abc) xyz \right] \le \\ &\le B \left[1 + k_1 abc \exp(k_1 abc) \right],\ \text{for a.e. } (x,y,z) \in D. \end{aligned} \tag{3.22}$$

This shows the saturated solution U^* is bounded on D.

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