LIBERATAS MATHEMATICA, Vol. 13 (1993)

A NONLINEAR BOUNDARY VALUE PROBLEM IN THE THEORY OF DEFLECTIONS OF SPHERICAL CAPS

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Abstract: The existence and qualitative properties of a second order nonlinear boundary value problem are studied.

1. Introduction

The nonlinear boundary value problem

$$\begin{aligned} &y'' = f(x,y) = \frac{-x^2}{32y^2} + \frac{\lambda^2}{8}, \quad 0 < x \leq 1 \\ &y(0) = 0, \quad 2y'(1) - (1+v)y(1) = 0, \quad \lambda > 0, \quad 0 < v < 1 \end{aligned} \tag{1.1}$$

arises in the large deflection membrane response of a spherical cap. This paper follows a study of this problem by Baxley [1] where an existence theory is presented, followed by some numerical results. Earlier results on this problem may be seen in Goldberg [2], Perrone and Kao [4] and Na [3]. In these papers, several numerical results are studied.

Two important questions concerning (1.1) are: i) existence of a unique positive solution in (0,1] and ii) the behavior of $\lim_{x \to 0} \frac{y}{x}$, the maximum radial stress, in terms of λ. In [1], the author establishes the existence of a positive soultion satisfying certain growth restrictions. This is accomplished by converting (1.1) into a nonlinear problem on the semi-infinite interval and applying earlier results for such class of nonlinear ordinary differential equations. An initial value problem-shooting approach is then used to obtain numerical solutions.

Our approach in this paper is to treat (1.1) directly as a boundary value problem. Eventhough, in this problem the boundary condition $y(0) = 0$ does not create any singularity. The methods of upper and lower solutions as utilized here can be easily adapted to handle more general singular nonlinear problems. For an extensive introduction to these concepts, refer to [5]. By using this approach we are able to establish the existence of a unique positive solution

(with no growth restrictions imposed). In [1], the author raises the question of evaluating $\lim\limits_{x\to 0} \frac{y(x)}{x}$ and conjectures that this limit is $\frac{1}{2\lambda}$. We answer this question negatively for certain values of λ and proves that for "large" λ, $\lim\limits_{x\to 0} \frac{y}{x}$ is essentially $\frac{1}{2\lambda}$. Our approach lends itself naturally to numerical solutions of (1.1), but we will discuss these aspects elsewhere.

2. Bounds for positive solutions of (1.1)

In this section we demonstrate the existence of positive upper and lower bounds for possible positive solutions of (1.1). By a positive upper solution β of (1) we mean a function β that satisfies

$$\beta > 0 \text{ on } (0,1],\ \beta'' \le f(x,\beta) \text{ on } [0,1]$$
$$\beta(0) \ge 0 \text{ and } 2\beta'(1) - (1+v)\beta(1) \ge 0.$$

A similar definition holds for positive lower solutions.

Lemma 2.1. $y_u(x) = \frac{x}{2\lambda}$ is a positive upper solution of (1.1).

Proof. It is clear that

$$y_u > 0 \text{ on } (0,\ 1],\ y_u(0) = 0,\ 2y_u'(1) - (1+v)y_u(1) \ge 0,$$

and

$$y_u'' = 0 = -\frac{x^2}{32y_u^2} + \frac{\lambda^2}{8}\ .$$

Thus, y_u is a positive upper solution of (1.1). This completes the proof.

Lemma 2.2. Let

$$A_l = \sqrt{\frac{\lambda^2}{\lambda^2 + \frac{8}{\lambda}}},\ B_l = \frac{A_l(1-v)}{2\lambda(3-v)},\ \text{and } y_l(x) = A_l \cdot \frac{x}{2\lambda} - B_l \cdot x^2.$$

Then y_l is a positive lower solution of (1.1).

Proof. It is clear that

$$y_l > 0 \text{ on } (0,\ 1],\ y_l(0) = 0,\ y_l'(\mathrm{x}) = \frac{A_l}{2\lambda} - 2B_l x,\ y_l''(x) = -2B_l,$$

and

$$2y_l'(1) - (1+v)y_l(1) = 2\left(\frac{A_l}{2\lambda} - 2B_l\right) - (1+v)\left(\frac{A_l}{2\lambda} - B_l\right) = (1-v)\frac{A_l}{2\lambda} - (3-v)B_l = 0.$$

From the definition of A_l and B_l, we have

$$A_l^2 \leq \frac{\lambda^2}{\lambda^2 + 16B_l} \text{ or } -2B_l \geq -\frac{\lambda^2}{8A_l^2} + \frac{\lambda^2}{8}. \tag{2.1}$$

From the inequality (2.1), we obtain

$$y_l'' = -2B_l \geq -\frac{1}{32\left(\frac{A_l}{2\lambda}\right)^2} + \frac{\lambda^2}{8} \geq -\frac{x^2}{32x^2\left(\frac{A_l}{2\lambda} - B_l x\right)^2} + \frac{\lambda^2}{8}.$$

Thus, y_l is a positive lower solution of (1.1). This completes the proof.

Lemma 2.3. Let y be a positive solution of (1.1) and let $\alpha,\ \beta \in C^2(\mathrm{I})$ be positive lower and positive upper solutions, respectively, of (1.1) with $\alpha(x) \leq \beta(x)$. Then $\alpha(x) \leq y(x) \leq \beta(x)$ on I.

Proof. Clearly, $y(0) \leq \beta(0)$. Suppose that there exists a $\xi \in (0,\ 1]$ such that $y(\xi) > \beta(\xi)$. By the continuity of $y - \beta$ and the fact that $y(0) \leq \beta(0)$, there exist δ_0 and the largest δ_1, $0 \leq \delta_0 < \delta_1 \leq 1$, such that $y(x) > \beta(x)$ on $(\delta_0,\ \delta_1)$ and $y(\delta_0) = \beta(\delta_0)$. Then, by the definition of a positive upper solution of (1.1) and the fact that $y > \beta$ on $(\delta_0,\ \delta_1)$, we have

$$y'' - \beta'' = -\frac{x_2}{32y^2} + \frac{\lambda^2}{8} - \beta'' \geq -\frac{x^2}{32\beta^2} + \frac{\lambda^2}{8} - \beta'' \geq 0 \text{ on } (\delta_0,\ \delta_1).$$

Thus, $y - \beta$ is convex on $[\delta_0,\ \delta_1]$. Since $y(\delta_0) = \beta(\delta_0)$ and $y - \beta$ is convex and positive on $(\delta_0,\ \delta_1)$, we have $\delta_1 = 1$ and $y(1) > \beta(1)$. Since $y - \beta$ is convex on $[\delta_0,$ $1]$, we know that

$$y'(1) - \beta'(1) \geq \frac{y(1) - \beta(1)}{1 - \delta_0} \geq y(1) - \beta(1). \tag{2.2}$$

From the inequality (2.2), we have

$$\frac{1+v}{2}y(1)-\frac{1+v}{2}\beta(1)\geq y'(1)-\beta'(1)\geq y(1)-\beta(1).$$

Since $0<v<1$ and $y(1)-\beta(1)>0$, we obtain

$$\frac{1+v}{2}y(1)-\frac{1+v}{2}\beta(1)<y(1)-\beta(1),$$

which is a contradiction. Therefore, $y(x)\leq\beta(x)$ on I.

Clearly, $\alpha(0)\leq y(0)$. Suppose that there exists a $\xi\in(0, 1]$ such that $\alpha(\xi)>y(\xi)$. By the continuity of $\alpha-y$ and the fact that $\alpha(0)\leq y(0)$, there exist δ_0 and the largest δ_1, $0\leq\delta_0<\delta_1\leq 1$, such that $\alpha(x)>y(x)$ on (δ_0, δ_1) and $\alpha(\delta_0)=y(\delta_0)$. Then, by the definition of a positive lower solution of (1.1) and the fact that $\alpha>y$ on (δ_0, δ_1), we have

$$\alpha''-y''=\alpha''+\frac{x^2}{32y^2}-\frac{\lambda^2}{8}\geq\alpha''+\frac{x^2}{32\alpha^2}-\frac{\lambda^2}{8}\geq 0 \text{ on } (\delta_0, \delta_1).$$

Thus, $\alpha-y$ is convex on $[\delta_0, \delta_1]$. Since $\alpha(\delta_0)=y(\delta_0)$ and $\alpha-y$ is convex and positive on (δ_0, δ_1), we have $\delta_1=1$ and $\alpha(1)>y(1)$. Since $\alpha-y$ is convex on $[\delta_0, 1]$, we know that

$$\alpha'(1)-y'(1)\geq\frac{\alpha(1)-y(1)}{1-\delta_0}\geq\alpha(1)-y(1). \tag{2.3}$$

From the inequality (2.3), we derive

$$\frac{1+v}{2}\alpha(1)-\frac{1+v}{2}y(1)\geq\alpha'(1)-y'(1)\geq\alpha(1)-y(1).$$

Since $0<v<1$ and $\alpha(1)-y(1)>0$, we have

$$\frac{1+v}{2}\alpha(1)-\frac{1+v}{2}y(1)<\alpha(1)-y(1),$$

which is a contradiction. Therefore, $\alpha(x)\leq y(x)$ on I. This completes the proof.

3. Existence of a positive solution for "perturbed" problems

We consider the nonlinear boundary value problem

$$(3.1)_m \qquad \begin{aligned} & y'' = -\frac{\lambda^2}{8}\frac{\left(\frac{x}{2}\lambda+\frac{1}{m}\right)^2}{y^2}+\frac{\lambda^2}{8}, \quad 0 \le x \le 1 \\ & y(0)=\frac{1}{m}, \quad 2y'(1)-(1+v)y(1)=0 \end{aligned}$$

which may be viewed as a perturbation of (1.1) .

Lemma 3.1. $y_{lm}(x)=\frac{1}{m}$ is a positive upper solution of $(3.1)_m$.

Proof. Clearly $2y'_{lm}(1)-(1+v)y_{lm}(1) \le 0$.

Also,

$$y''_{lm} \ge -\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m}\right)^2}{y^2_{lm}}+\frac{\lambda^2}{8}.$$

Thus, y_{lm} is a positive lower solution of $(3.1)_m$. This completes the proof.

Lemma 3.2. Let $m \ge \frac{2\lambda(1+v)}{1-v}$. Then $y_{um}(x)=\frac{x}{2\lambda}+\frac{1}{m}$ is a positive upper solution of $(3.1)_m$.

Proof. It is obvious that $y_{um}>0$ on (0, 1], $y_{um}(0)=\frac{1}{m}$, $y'_{um}=\frac{1}{2\lambda}$, and $y''_{um}=0$.

From the assumption on m, we know that $\frac{1}{m} \le \frac{1-v}{2\lambda(1+v)}$. So, we obtain

$$2y'_{um}(1)-(1+v)y_{um}(1) \ge 0 \text{ and } y''_{um} \le -\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m}\right)^2}{y^2_{um}}+\frac{\lambda^2}{8}.$$

Thus, y_{um} is a positive upper solution of $(3.1)_m$. This completes the proof.

Lemma 3.3. If $m \ge \frac{2\lambda(1+v)}{1-v}$, then there exists a solution y_m of $(3.1)_m$ such that

$$y_{lm}(x) \le y_m(x) \le y_{um}(x) \text{ on I},$$

where y_{um} and y_{lm} are given in Lemma 3.1 and Lemma 3.2.

Proof. It is obvious that

$$f(x,y) = -\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m}\right)^2}{y^2}+\frac{\lambda^2}{8}$$

is bounded and continuous on w, where $w=\Big\{(x,\ y)\ :\ y_{lm}(x)\le y\le y_{um}(x),\ x\in \mathrm{I}\Big\}$. It is easy to see, as in Lemma 2.3, that any solution y_m of $(3.1)_m$ satisfies $y_{lm}\le y_m\le y_{um}$. An application of Schauder's theorem establishes the existence of a solution.

Now we want to show the uniqueness of a positive solution of $(3.1)_m$.

Lemma 3.4. If y_1 and y_2 are two positive solutions of $(3.1)_m$, then $y_1\equiv y_2$.

Proof. Let y_1 and y_2 be positive solutions of $(3.1)_m$. Then we obtain

$$y_1''-y_2''=\frac{x^2}{32y_1^2\cdot y_2^2}\ (y_1^2-y_2^2)\ \text{on}\ (0,\ 1). \tag{3.1}$$

If we multiply both sides of (3.1) by (y_1-y_2), then we have

$$(y_1''-y_2'')(y_1-y_2)=\frac{x^2}{32y_1^2\cdot y_2^2}\ (y_1+y_2)(y_1-y_2)^2\ \ge 0\ \text{on}\ (0,\ 1). \tag{3.2}$$

Therefore, if we integrate both sides of (3.2) from 0 to 1, then we obtain

$$\begin{aligned}
&0\le\int_0^1 (y_1''-y_2'')(y_1-y_2)dx,\\
&0\le(y_1'(1)-y_2'(1))(y_1(1)-y_2(1))-\int_0^1 (y_1'-y_2')^2dx,\\
&0\le\frac{1+v}{2}(y_1(1)-y_2(1))^2-\int_0^1 (y_1'-y_2')^2dx.
\end{aligned} \tag{3.3}$$

From (3.3), we derive

$$\frac{1+v}{2}(y_1(1)-y_2(1))^2\ge\int_0^1 (y_1'-y_2')^2dx.$$

Now, if $y_1(1)-y_2(1)\ne 0$, then

$$0<\frac{1+v}{2}\ (y_1(1)-y_2(1))^2<(y_1(1)-y_2(1))^2\le\Big(\int_0^1 (y_1'-y_2')dx\Big)^2\le\int_0^1 (y_1'-y_2')^2dx$$

and so,
$$\frac{1+v}{2}\,(y_1(1)-y_2(1))^2 < \int_0^1 (y_1'-y_2')^2 dx.$$

This is a contradiction. Therefore, we have

$$y_1(1)=y_2(1) \text{ and } \int_0^1 (y_1'-y_2')^2 dx \le 0 \text{ from the inequality (3.3).}$$

Thus, we obtain

$$\int_0^1 (y_1'-y_2')^2 dx = 0,$$

which implies that $y_1'-y_2'=0$ and $y_1-y_2=\text{constant}$. Since $y_1(0)=y_2(0)$ and $y_1(1)=y_2(1)$, we have $y_1\equiv y_2$. This completes the proof.

Lemma 3.5. If $m \ge \frac{2\lambda(1+v)}{1-v}$, y_m is a positive solution of $(3.1)_m$, and $l(x)$ is a positive lower solution of (1.1), then $l(x)\le y_m(x)$ on **I**.

Proof. Since $l(x)$ is a positive lower solution of (1.1), we have

$$l'' \ge -\frac{x^2}{32l^2}+\frac{\lambda^2}{8} = -\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}\right)^2}{l^2}+\frac{\lambda^2}{8} \ge -\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m}\right)^2}{l^2}+\frac{\lambda^2}{8}.$$

Thus, l is a lower solution of $(3.1)_m$ and by Lemma 3.4, $l(x)\le y_m(x)$ on **I**. This completes the proof.

Note. From Lemma 3.5, we know that all positive solutions of $(3.1)_m$, $m\ge\frac{2\lambda(1+v)}{1-v}$, are bounded below by a positive lower solution of (1.1) on **I**.

Lemma 3.6. If $m_1 \ge m_2 \ge \frac{2\lambda(1+v)}{1-v}$ and y_{m_1} and y_{m_2} are positive solutions of $(3.1)_{m_1}$ and $(3.1)_{m_2}$, respectively, then $y_{m_1}\le y_{m_2}$ on **I**.

Proof. It is clear that

$$y_{m_2}(0)=\frac{1}{m_2}\ge\frac{1}{m_1} \text{ and } 2y_{m_2}'(1)-(1+v)y_{m_2}(1)=0.$$

Also, we have

$$y''_{m_2} = -\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m_2}\right)^2}{y^2_{m_2}}+\frac{\lambda^2}{8} \leq -\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m_1}\right)^2}{y^2_{m_2}}+\frac{\lambda^2}{8}.$$

So, y_{m_2} is an upper solution of $(3.1)_{m_1}$ and hence, $y_{m_1} \leq y_{m_2}$ on I. This completes the proof.

Remark. From Lemma 3.6 it follows that there exists a monotone decreasing sequence of functions $\{y_m\}$ defined on I. We will use this sequence to show the existence of a positive solution of (1.1).

4. Existence of a solution for (1.1)

Theorem 4.1. (Existence) Let m_0 be a positive integer such that $m_0 \geq \frac{2\lambda(1+v)}{1-v}$. If y_{m_0+m} is a positive solution of $(3.1)_{m_0+m}$ for each $m = 1, 2, 3, \ldots$, then the sequence $\{y_{m_0+m}\}$ converges to a positive solution y of (1.1).

Proof. To prove this theorem, we prove the following steps:

Step 1: $y_{m_0+m} \longrightarrow y$ as $m \longrightarrow \infty$.

Step 2: $y \in C^2((0, 1])$.

Step 3: y is a solution of (1.1).

Step 1: From Lemma 3.5 and Lemma 3.6, we know that the sequence $\{y_{m_0+m}\}$ is monotone decreasing in m and is bounded below by a positive lower solution $l(x)$ of (1.1), where $l(x)$ is given in Lemma 2.1. Therefore,

$$y_{m_0+m} \longrightarrow y \text{ as } m \longrightarrow \infty \text{ and } y(x) \geq l(x) \text{ on I}.$$

Step 2: If we integrate y''_{m_0+m} from x to 1, then we have

$$-y'_{m_0+m}(x) + y'_{m_0+m}(1) = \int_x^1 \left(-\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}}+\frac{\lambda^2}{8}\right) dx,$$

$$y'_{m_0+m}(x) = y'_{m_0+m}(1) + \int_x^1 \left(\frac{\lambda^2}{8}\cdot\frac{\left(\frac{x}{2\lambda}+\frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}}-\frac{\lambda^2}{8}\right) dx, \tag{4.1}$$

and

$$y'_{m_0+m}\left(\tfrac{1}{2}\right) = \frac{1+v}{2}\, y_{m_0+m}(1) + \int_{\frac{1}{2}}^{1} \left(\frac{\lambda^2}{8} \cdot \frac{\left(\frac{x}{2\lambda} + \frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}} - \frac{\lambda^2}{8} \right) dx. \tag{4.2}$$

If we let $m \longrightarrow \infty$ in both sides of (4.2), then, by Lebesgue's Dominated Convergence Theorem, we obtain $y'_{m_0+m}\left(\tfrac{1}{2}\right) \longrightarrow b$ as $m \longrightarrow \infty$. Now, from (4.1) and (4.2), we derive

$$\begin{aligned} y'_{m_0+m}(x) - y'_{m_0+m}\left(\tfrac{1}{2}\right) &= \int_x^1 \left(\frac{\lambda^2}{8} \cdot \frac{\left(\frac{x}{2\lambda} + \frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}} - \frac{\lambda^2}{8} \right) dx - \int_{\frac{1}{2}}^{1} \left(\frac{\lambda^2}{8} \cdot \frac{\left(\frac{x}{2\lambda} + \frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}} - \frac{\lambda^2}{8} \right) dx \\ &= \frac{\lambda^2}{8} \int_x^{\frac{1}{2}} \frac{\left(\frac{x}{2\lambda} + \frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}}\, dx - \frac{\lambda^2}{8}\left(\frac{1}{2} - x\right). \end{aligned} \tag{4.3}$$

If we integrate both sides of (4.3) from $\frac{1}{2}$ to x, then we obtain

$$\begin{aligned} & y_{m_0+m}(x) - y_{m_0+m}\left(\tfrac{1}{2}\right) - y'_{m_0+m}\left(\tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\right) \\ &= \frac{\lambda^2}{8} \int_{\frac{1}{2}}^{x} \int_{s}^{\frac{1}{2}} \frac{\left(\frac{x}{2\lambda} + \frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}}\, ds\, dx - \frac{\lambda^2}{8} \int_{\frac{1}{2}}^{x} \left(\frac{1}{2} - x\right) dx \\ &= \frac{\lambda^2}{8} \int_x^{\frac{1}{2}} (x-t) \cdot \frac{\left(\frac{t}{2\lambda} + \frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}}\, dt - \frac{\lambda^2}{8} \int_{\frac{1}{2}}^{x} \left(\frac{1}{2} - x\right) dx. \end{aligned} \tag{4.4}$$

Let $0 < \epsilon < \frac{1}{2}$ and $x \in [\epsilon, 1]$. Then we obtain

$$\frac{\left(\frac{x}{2\lambda} + \frac{1}{m_0+m}\right)^2}{y^2_{m_0+m}} \longrightarrow \frac{\left(\frac{x}{2\lambda}\right)^2}{y^2} \text{ on } [\epsilon,\ 1] \text{ as } m \longrightarrow \infty.$$

So, if we let $m \longrightarrow \infty$ in both sides of (4.4), then, by Lebesgue's Dominated Convergence Theorem, we obtain

$$y(x) - y\left(\tfrac{1}{2}\right) - b\left(x - \tfrac{1}{2}\right) = \frac{\lambda^2}{8}\int_{x}^{\frac{1}{2}} (x-t)\cdot\frac{\left(\frac{t}{2\lambda}\right)^2}{y^2}\,dt - \frac{\lambda^2}{8}\int_{\frac{1}{2}}^{x}\left(\tfrac{1}{2} - x\right)\,dx, \tag{4.5}$$

which implies $y \in C^2((0,\ 1])$.

Step 3: Now, we show that y is a positive solution of (1.1). It is obvious that

$$y(0) = 0 \text{ and } y(x) > 0 \text{ on } (0,\ 1].$$

From the construction of y, we obtain $\lim_{x\to 0+} y(x) = 0$ which implies that y is continuous at 0. If we take the first and second derivatives of both sides of (4.5), then we obtain

$$y'(x) - b = \frac{\lambda^2}{8}\left(\int_{x}^{\frac{1}{2}} \frac{\left(\frac{t}{2\lambda}\right)^2}{y^2}\,dt - \left(\tfrac{1}{2} - x\right)\right) \tag{4.6}$$

and

$$y''(x) = \frac{\lambda^2}{8}\left(-\frac{\left(\frac{x}{2\lambda}\right)^2}{y^2} + 1\right) = -\frac{x^2}{32y^2} + \frac{\lambda^2}{8}. \tag{4.7}$$

Also, we obtain

$$\frac{1+v}{2}\,y(1) = \lim_{m\to\infty}\frac{1+v}{2}\,y_{m_0+m}(1) = \lim_{m\to\infty} y'_{m_0+m}(1) = b + \frac{\lambda^2}{8}\int_{1}^{\frac{1}{2}}\frac{\left(\frac{x}{2\lambda}\right)^2}{y^2}\,dx + \frac{\lambda^2}{8}\cdot\frac{1}{2}.$$

If we integrate both sides of (4.7) from $\frac{1}{2}$ to 1, then

$$y'(1) - y'\left(\tfrac{1}{2}\right) = \frac{\lambda^2}{8}\int_{1}^{\frac{1}{2}}\frac{\left(\frac{x}{2\lambda}\right)^2}{y^2}\,dx + \frac{\lambda^2}{8}\cdot\frac{1}{2},$$

which implies that

$$\frac{1+v}{2}\,y(1) = y'(1).$$

Thus, y is a positive solution of (1.1). This completes the proof.

Corollary 4.2. Let y be a positive solution of (1.1). Then $\frac{y}{x}$ is nonincreasing on (0, 1].

Proof. By Lemma 2.1 and Lemma 2.3, $y(x) \le \frac{x}{2\lambda}$ on I and $y'' \le 0$ on (0, 1). Therefore, $-y$ is convex on I and thus, $-\frac{y}{x}$ is nondecreasing on (0, 1], which means that $\frac{y}{x}$ is nonincreasing on (0, 1].

5. Uniqueness of a positive solution of (1.1)

Theorem 5.1. Assume that y_1 and y_2 are positive solution of (1.1). Then $y_1 \equiv y_2$.

Proof. The proof is similar to the proof of Lemma 3.4.

Let y_1 and y_2 be positive solutions of (1.1). Then

$$y_1'' - y_2'' = \frac{x^2}{32 y_1^2 \cdot y_2^2}\,(y_1^2 - y_2^2) \text{ on } (0,\ 1). \tag{5.1}$$

So, if we multiply both sides of (5.1) by $y_1 - y_2$, then we obtain

$$(y_1'' - y_2'')(y_1 - y_2) = \frac{x^2}{32 y_1^2 \cdot y_2^2}\,(y_1 + y_2)(y_1 - y_2)^2 \ \ge 0 \text{ on } (0,\ 1). \tag{5.2}$$

Therefore, if we integrate both sides of (5.2) from δ to 1, then

$$0 \le \int_\delta^1 (y_1'' - y_2'')(y_1 - y_2)\,dx,$$

$$0 \le (y_1'(1) - y_2'(1))(y_1(1) - y_2(1)) - (y_1'(\delta) - y_2'(\delta))(y_1(\delta) - y_2(\delta)) - \int_\delta^1 (y_1' - y_2')^2\,dx,$$

and

$$0 \le \frac{1+v}{2}(y_1(1) - y_2(1))^2 - (y_1'(\delta) - y_2'(\delta))(y_1(\delta) - y_2(\delta)) - \int_\delta^1 (y_1' - y_2')^2\,dx. \tag{5.3}$$

By Lemma 2.1, Lemma 2.2, and Lemma 2.3, $|y_i'(x)|$ is bounded on (0, h) for small enough h. Now, if we let $\delta \longrightarrow 0+$ in both sides of (5.3), then we obtain

$$0 \le \frac{1+v}{2}(y_1(1) - y_2(1))^2 - \int_0^1 (y_1' - y_2')^2\,dx, \tag{5.4}$$

which implies that

$$\frac{1+v}{2}(y_1(1)-y_2(1))^2 \ge \int_0^1 (y_1'-y_2')^2\,dx .$$

Now, if $y_1(1)-y_2(1)\neq 0$, then

$$0<\frac{1+v}{2}\,(y_1(1)-y_2(1))^2<(y_1(1)-y_2(1))^2\le\Big(\int_0^1 (y_1'-y_2')\,dx\Big)^2\le\int_0^1 (y_1'-y_2')^2\,dx$$

and hence,

$$\frac{1+v}{2}\,(y_1(1)-y_2(1))^2<\int_0^1 (y_1'-y_2')^2\,dx .$$

This is a contradiction. Therefore, we have

$$y_1(1)=y_2(1) \text{ and } \int_0^1 (y_1'-y_2')^2\,dx\le 0 \text{ from } (5.4).$$

Thus, we obtain

$$\int_0^1 (y_1'-y_2')^2\,dx=0.$$

which means that $y_1'-y_2'=0$ and $y_1-y_2=\text{constant}$. Since $y_1(0)=y_2(0)$ and $y_1(1)=y_2(1)$, we obtain $y_1\equiv y_2$. This completes the proof.

Remark. Baxley [1] proves that there exists at most one positive solution $y(x)$ satisfying

$$\frac{y(x)}{x^p}\longrightarrow 0 \text{ as } x\longrightarrow 0+$$

where $\frac{1+v}{2}<p<1$. Also, he leaves open the possibility of multiple positive solutions satisfying $y(x)\longrightarrow 0$ as $x\longrightarrow 0+$ more slowly than $x^{\frac{1+v}{2}}$. However, from Theorem 5.1 we know that there exists only one positive solution and hence there is no positive solution satisfying $y(x)\longrightarrow 0$ as $x\longrightarrow 0+$ more slowly than $x^{\frac{1+v}{2}}$.

6. The behavior of $\lim\limits_{x \to 0+} \frac{y(x)}{x}$

In this section, we will consider Baxley's second question in [1]. To answer this question, we will use Lemma 2.3. For certain values of λ, we will construct a positive upper solution of (1.1) which is less than $y = \frac{x}{2\lambda}$. Now, we consider $y = A\frac{x}{2\lambda} - Bx^2$, $0 < A < 1$, $0 < B$, and we find the conditions on A and B so that y is a positive upper solution of (1.1). To satisfy $0 < y$ on $(0, 1]$, we have

$$0 < \frac{A}{2\lambda} - Bx \text{ on } (0,\ 1]. \tag{6.1}$$

If $2\lambda B < A$, then the inequality (6.2) is satisfied. To satisfy the boundary condition for a positive upper solution of (1.1), we have

$$0 \leq 2\left(\frac{A}{2\lambda} - 2B\right) - (1+v)\left(\frac{A}{2\lambda} - B\right) \text{ or } 0 \leq (1-v)\frac{A}{2\lambda} - (3-v)B. \tag{6.2}$$

So, if $A \leq 2\lambda\left(\frac{3-v}{1-v}\right)B$, then the inequality (6.2) is satisfied. To satisfy the condition for a positive upper solution of (1.1), we have

$$-2B \leq -\frac{x^2}{32x^2\left(\frac{A}{2\lambda} - Bx\right)^2} + \frac{\lambda^2}{8} \text{ or } -2B \leq -\frac{1}{32\left(\frac{A}{2\lambda} - Bx\right)^2} + \frac{\lambda^2}{8}. \tag{6.3}$$

It is clear that

$$-\frac{1}{32\left(\frac{A}{2\lambda} - B\right)^2} \leq -\frac{1}{32\left(\frac{A}{2\lambda} - Bx\right)^2}.$$

So, if

$$\left(\frac{A}{2\lambda} - B\right)^2 \geq \frac{1}{32\left(2B + \frac{\lambda^2}{8}\right)} \text{ or } A \geq 2\lambda B + \frac{2\lambda}{\sqrt{32\left(2B + \frac{\lambda^2}{8}\right)}},$$

then the inequality (6.3) is satisfied. Now, we consider the equation

$$A(B) = 2\lambda B + \frac{2\lambda}{\sqrt{32\left(2B + \frac{\lambda^2}{8}\right)}}.$$

Then we know that

$$A(0) = 1, \ \frac{dA}{dB}(0) = 2\lambda\left(1 - \frac{4}{\lambda^3}\right), \text{ and } \frac{dA}{dB}(B) \geq \frac{dA}{dB}(0).$$

To get $A < 1$, we need $\frac{dA}{dB}(0) < 0$ and so, $\lambda^3 < 4$ or $\lambda < 4^{\frac{1}{3}}$. Now, we solve

$$A(B) = 2\lambda B + \frac{2\lambda}{\sqrt{32(2B + \frac{\lambda^2}{8})}} = 1 \text{ for } B.$$

Then we obtain

$$B = 0 \text{ or } B_{\pm} = \frac{-(\lambda^2 - \frac{16}{\lambda}) \pm \sqrt{\lambda^4 + 32\lambda}}{32}.$$

Clearly, we know that if we replace B with B_+ in $A(B)$ then $A(B_+) \neq 1$ and so B_+ is not a solution of $A(B) = 1$.

Summarizing the above discussion yields the following result.

Lemma 6.1. Let

$$\lambda^3 < 4, \ B_0 = -\frac{1}{32}\left(\lambda^2 - \frac{16}{\lambda} + \sqrt{\lambda^4 + 32\lambda}\ \right), \ B_u = \frac{1}{2}\ \mathrm{min}\left(B_0, \ \frac{1-v}{2\lambda(3-v)}\right),$$

and let

$$A_u = 2\lambda B_u + 2\lambda\left(64B_u + 4\lambda^2\right)^{-\frac{1}{2}}.$$

Then $y_u = A_u \cdot \frac{x}{2\lambda} - B_u \cdot x^2$ is a positive upper solution of (1.1).

Theorem 6.2. Let $\lambda^3 < 4$ and let y be a positive solution of (1.1). Then we have

$$\lim_{x \to 0+} \frac{y}{x} < \frac{1}{2\lambda}.$$

Proof. It is clear from Lemma 6.1 and Lemma 2.3.

Note. Theorem 6.2 answers Baxley's second question in [1] in part, but Baxley's second question for large λ is still left open. However, we know from Lemma 2.2 that

$$A_l \longrightarrow 1 \text{ as } \lambda \longrightarrow \infty.$$

So we expect that $\lim\limits_{x \to \infty} \frac{y}{x} \approx \frac{1}{2\lambda}$ for large enough λ. Also, our numerical results show that $\lim\limits_{x \to 0+} \frac{y}{x} \approx \frac{1}{2\lambda}$ for large enough λ.

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