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NON – LINEAR BOUNDARY VALUE PROBLEMS INVOLVING ABSTRACT VOLTERRA OPERATORS

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Introduction

In this paper we shall investigate the existence of solutions of some non-linear boundary value problems for functional differential equations involving abstract Volterra operators.

Consider the boundary value problem

$$\dot{x}(t) = (Lx)(t) + (fx)(t), \qquad t \in [0, T], \tag{1.1}$$

$$A\, x(0) + B\, x(T) = hx \in \mathbb{R}^n, \tag{1.2}$$

where A and B are given $n \times n$ matrices with real entries, $L : L^p([0, T], \mathbb{R}^n) \to L^p([0, T], \mathbb{R}^n)$, $h : C([0, T], \mathbb{R}^n) \to \mathbb{R}^n$, and $f : C([0, T], \mathbb{R}^n) \to L^p([0, T], \mathbb{R}^n)$, $1 < p < \infty$.

1. Existence

In the paper [8] we obtained the integral representation of the unique solution of $\dot{x}(t) = (Lx)(t) + f(t)$, with $x(0) = x_0$, under the main assumption that L is a linear continuous operator of Volterra type; namely,

$$x(t) = C(t)\, x_0 + \int_0^t R(t,s)\, f(s)\, ds. \tag{1.3}$$

If we replace $f(t)$ by $(fx)(t)$ we obtain

$$x(t) = C(t)\, x_0 + \int_0^t R(t,s)\, (fx)(s)\, ds, \tag{1.4}$$

which is easily seen to be equivalent to (1.1), under initial condition $x(0) = x_0$.

Now, if we substitute $x(t)$ from(1.4) in (1.2), we obtain the initial value that produces the solution verifying (1.2) :

$$[\, A + B\, C(T)\,]\, x_0 = hx - B \int_0^T R(T,s)\, (fx)(s)\, ds.$$

If $\det [A + B\,C(T)] \neq 0$, then

$$x_0 = [A + B\,C(T)]^{-1} \left\{ hx - B \int_0^T R(T,s)\,(fx)(s)\,ds \right\}. \tag{1.5}$$

Now substituting (1.5) in (1.4) we obtain

$$x(t) = (gx)(t) + \int_0^T \tilde{R}(t,s)\,(fx)(s)\,ds \tag{1.6}$$

where

$$(gx)(t) = C(t)\,[A + B\,C(T)]^{-1}\,(hx), \tag{1.7}$$

and

$$\tilde{R}(t,s) = \begin{cases} R(t,s) - C(t)\,[A + B\,C(T)]^{-1}\,B\,R(T,s), & 0 \leq s \leq t. \\ -C(t)\,[A + B\,C(T)]^{-1}\,B\,R(T,s), & t \leq s \leq T. \end{cases}$$

Notice that we have reduced the boundary value problem (1.1), (1.2) to a non-linear Hammerstein integral equation (1.6). See [1,5,7], regarding this type of equation.

In the classical theory of Hammerstein equations, f is usually an operator of Niemytski type (for definition see [6]). While this remains a valid choice, it is not necessary to limit our investigation to this particular case.

Assume the following conditions are satisfied with respect to the boundary value problem (1.1), (1.2):

(i) $L : L^p([0, T], \mathbb{R}^n) \to L^p([0, T], \mathbb{R}^n)$ is a linear continuous operator of Volterra type, $1 < p < \infty$.

(ii) $f : C ([0, T], \mathbb{R}^n) \to L^p([0, T], \mathbb{R}^n)$ is a continuous operator such that the inequality

$$| gx |_C + \gamma\, \phi(\rho) \leq \rho$$

has a positive solution in ρ, where $\phi(\rho)$ is a positive nondecreasing function for $\rho > 0$, such that

$$\sup_{|x|_C \leq \rho} | fx |_{L^p} = \phi(\rho) < \infty,$$

and $\gamma = M_1^{\frac{1}{q}} > 0$. M_1 will be fixed later.

(iii) $\det [A + B\,C(T)] \neq 0$, where $C(t)$ is as in [8].

(iv) $h : C([0, T], \mathbf{R}^n) \to \mathbf{R}^n$ satisfies the Lipschitz condition

$$|hx - hy| \leq \alpha \, |x - y|_C, \qquad \forall\, x, y \in C([0, T], \mathbf{R}^n),$$

with α small enough.

Theorem 1 Under the above assumptions, there exists at least one solution $x(t)$ of (1.1),(1.2) satisfying (1.1) a.e. on $[0, T]$, and such that $|x|_C \leq \rho$.

Proof: In order to obtain an existence result, we shall concentrate on the equation (1.6). The equation (1.6) can be written as

$$x(t) = (Ux)(t),$$

where

$$(Ux)(t) = (gx)(t) + \int_0^T \tilde{R}(t,s)\,(fx)(s)\,ds. \tag{1.8}$$

Since the right hand side of (1.8) is nothing else but the right hand side of (1.1.3), with $f(s)$ replaced by $(fx)(s)$ and x_0 replaced by (1.7), there results that $x \to Ux$ is an operator from $C([0, T], \mathbf{R}^n)$ into itself.

In view of the application of the Schauder fixed point theorem to the operator U defined by (1.8), we need first to show the continuity of this operator on the space $C([0, T], \mathbf{R}^n)$.

From (1.8) we have, in the Euclidean norm,

$$|(Ux)(t) - (Uy)(t)| \leq |C(t)[A + B\,C(T)]^{-1}|\,|(hx)(t) - (hy)(t)| + \int_0^T |\tilde{R}(t,s)|\,|(fx)(s) - (fy)(s)|\,ds.$$

Since h satisfies the Lipschitz condition and $C(t)$ is absolutely continuous on $[0, T]$,

$$|(Ux)(t) - (Uy)(t)| \leq \alpha k\,|x - y|_C + \int_0^T |\tilde{R}(t,s)|\,|(fx)(s) - (fy)(s)|\,ds, \tag{1.9}$$

where $k > 0$ is a constant.

In the paper [8] we obtained the following property

$$\int_0^t |R(t,s)|^q \, ds \le M, \qquad \text{for any} \quad t \in [0, T],$$

for the kernel $R(t, s)$ of the integral representation of solution. From this property we have

$$\int_0^T |\tilde{R}(t,s)|^q \, ds \le M_1, \qquad t \in [0, T]. \tag{1.10}$$

Applying the Holder inequality to (1.9), and using (1.10), we obtain

$$|(Ux)(t) - (Uy)(t)| \le \alpha k \, |x-y|_C + M_1^{\frac{1}{q}} \left(\int_0^T |(fx)(s) - (fy)(s)|^p \, ds \right)^{\frac{1}{p}}.$$

Hence,

$$|Ux - Uy|_C \le \alpha k \, |x-y|_C + M_1^{\frac{1}{q}} \, |fx - fy|_{L^p}, \tag{1.11}$$

which implies the continuity of U on $C([0, T], \mathbb{R}^n)$, on behalf of (ii).

The next property we have to prove for the operator U is compactness (in the sense of the topology of the space $C([0, T], \mathbb{R}^n)$). This means we have to show U is bounded on the space $C([0, T], \mathbb{R}^n)$, and that the set $\left\{ V : V = Ux, \; |x|_C \le \rho \right\}$ is equicontinuous on $[0, T]$. From (1.8) we have

$$|(Ux)(t)| \le |(gx)(t)| + \int_0^T |\tilde{R}(t,s)| \, |(fx)(s)| \, ds. \tag{1.12}$$

Applying the Holder inequality to (1.12), by taking the supremum of both sides over $t \in [0, T]$, and letting $\gamma = M_1^{\frac{1}{q}}$,

$$\begin{aligned} |Ux|_C &\le |gx|_C + \gamma \, |fx|_{L^p} \\ &\le |gx|_C + \gamma \sup_{|x|_C \le \rho} |fx|_{L^p}. \end{aligned}$$

Therefore,

$$|V|_C \le |gx|_C + \gamma \, \phi(\rho). \tag{1.13}$$

The inequality (1.13) implies the boundedness of the operator U on $C([0, T], \mathbb{R}^n)$ and since $C([0, T], \mathbb{R}^n)$ is a subspace of $L^p([0, T], \mathbb{R}^n)$, the set $\{ V : V = Ux, \ |x|_C \le \rho \}$ is bounded in $L^p([0, T], \mathbb{R}^n)$. Consequently, the set

$$\{ LV : V = Ux, \ |x|_C \le \rho \}$$

is bounded in L^p (L is a bounded operator), i.e., $|LV|_{L^p}$ remains bounded when $|x|_C \le \rho$.

But

$$V'(t) = (LV)(t) + (fx)(t),$$

which means that the set $\{ V'(t) : V = Ux, \ |x|_C \le \rho \}$ is bounded in $L^p([0, T], \mathbb{R}^n)$. This property immediately implies the equicontinuity of the set $\{ V : V = Ux, \ |x|_C \le \rho \}$ on $[0, T]$. Indeed we have

$$|V(t) - V(s)| \le \int_s^t |V'(u)| \, du. \tag{1.14}$$

Applying the Holder inequality to (1.14), we get

$$|V(t) - V(s)| \le |t - s|^{\frac{1}{q}} \left(\int_0^T |V'(u)|^p \, du \right)^{\frac{1}{p}}, \qquad 1 < p < \infty.$$

which implies the equicontinuity of the set $\{ V : V = Ux, \ |x|_C \le \rho \}$ on $[0, T]$. Therefore, by Arzela - Ascoli theorem, U is a compact operator.

Finally we need to show that U takes a closed convex set into itself. From the estimate (1.13) we have

$$|V|_C = |Ux|_C \le |gx|_C + \gamma \, \phi(\rho) \qquad \text{for} \qquad |x|_C \le \rho$$

and by assumption (ii) of the theorem we get

$$|Ux|_C \le \rho \qquad \text{for} \qquad |x|_C \le \rho,$$

which implies that the ball of radius $\rho > 0$, centered at the origin of the space $C([0, T], \mathbb{R}^n)$ is taken into itself by the operator U. Therefore, by Schauder fixed point theorem there exists at least one fixed point for U, i.e. $Ux = x$. Hence, there exists at least one solution $x(t)$ of (1.1), (1.2), satisfying (1.1) a.e. on $[0, T]$, and such that $|x|_C \leq \rho$, $\rho > 0$ being as in (ii).

Remark 1: In the proof of the continuity of U, we have obtained the inequality

$$|Ux - Uy|_C \leq \alpha k \; |x - y|_C + \gamma \; |fx - fy|_{L^p} .$$

If f satisfies a Lipschitz condition

$$|fx - fy|_{L^p} \leq \lambda \; |x - y|_C \qquad \forall\, x, y \in C([0, T], \mathbb{R}^n),$$

with λ small enough, then

$$|Ux - Uy|_C \leq \alpha k \; |x - y|_C + \gamma \lambda \; |x - y|_C ,$$

thus

$$|Ux - Uy|_C \leq (\alpha k + \gamma \lambda) \; |x - y|_C .$$

Now if α and λ are small enough, such that $\alpha k + \gamma \lambda < 1$, then U is a contraction on $C([0, T], \mathbb{R}^n)$ and hence by Banach contraction theorem U has a unique fixed point, i.e. $Ux = x$. Therefore there exists a unique solution, $x(t)$, in $C([0, T], \mathbb{R}^n)$, of the problem (1.1), (1.2).

Another existence result can be obtained by using some monotone operators arguments (see H. Brezis and F. Browder [2]).

2. Further existence results

Consider the system

$$\dot{x}(t) = (Lx)(t) + f(t, x(t)), \tag{2.1}$$

under the boundary value condition

$$A\, x(0) + B\, x(T) = \theta, \tag{2.2}$$

where θ is the zero vector in $\mathbb{R}^n$. Reduce (2.1), (2.2), as before, to the following non-linear Hammerstein integral equation

$$x(t) = \int_0^T \tilde{R}(t,s)\, f(s, x(s))\, ds. \tag{2.3}$$

Now (2.3) can be written as

$$x(t) = (Hx)(t),$$

where

$$(Hx)(t) = \int_0^T \tilde{R}(t,s)\, f(s, x(s))\, ds, \qquad t \in [0, T], \tag{2.4}$$

is the Hammerstein integral operator which appears as the product of a linear integral operator and a Niemytski operator, that is,

$$H = \tilde{L}\, F,$$

where F is given by

$$(Fx)(t) = f(t, x(t)), \tag{2.5}$$

and $\tilde{L}$ stands for

$$(\tilde{L}x)(t) = \int_0^T \tilde{R}(t,s)\, x(s)\, ds. \tag{2.6}$$

We intend to show that $\tilde{L}$ is a compact operator and then use this property to obtain an existence theorem for (2.3). From (2.6) we have

$$|(\tilde{L}x)(t)| \leq \int_0^t |R(t,s)\, x(s)|\, ds + \int_0^t |C(t)\,[A + B\, C(T)\,]^{-1}\, B|\; |R(T,s)\, x(s)|\, ds + \int_t^T |C(t)\,[A + B\, C(T)\,]^{-1}\, B|\; |R(T,s)\, x(s)|\, ds.$$

Since $C(t)$ is absolutely continuous on $[0, T]$, we have

$$|C(t)\,[A + B\, C(T)\,]^{-1}\, B| \leq \bar{k}$$

for some $\bar{k}$.

Therefore,

$$|(\tilde{L}x)(t)| \leq \int_0^T |R(t,s)\, x(s)|\, ds + \bar{k} \int_0^T |R(T,s)\, x(s)|\, ds. \tag{2.7}$$

Applying the Holder inequality to (2.7), and using

$$\int_0^T |R(t,s)|^q\, ds \leq M,$$

$$|(\tilde{L}x)(t)| \leq M^{\frac{1}{q}} \left(\int_0^T |x(s)|^p \, ds \right)^{\frac{1}{p}} + \bar{k}\, M^{\frac{1}{q}} \left(\int_0^T |x(s)|^p \, ds \right)^{\frac{1}{p}},$$

letting $k = M^{\frac{1}{q}} + \bar{k}\, M^{\frac{1}{q}}$,

$$|(\tilde{L}x)(t)| \leq k \left(\int_0^T |x(s)|^p \, ds \right)^{\frac{1}{p}}.$$

The above inequality implies that $\tilde{L}$ is bounded on $L^p([0, T], \mathbb{R}^n)$.

Now we show the equicontinuity of the set

$$\left\{ (\tilde{L}x)(t) : \; x \in L^p([0, T], \mathbb{R}^n), \; |x|_{L^p} \leq M_2 \,, \; M_2 > 0 \right\}.$$

For each $t_0 \in [0, T]$ we have

$$|(\tilde{L}x)(t) - (\tilde{L}x)(t_0)| \leq \int_0^t |R(t,s) - R(t_0, s)| \; |x(s)| \, ds + \int_0^t |C(t_0) - C(t)| \; |[A + B\,C(T)]^{-1} B| \; |R(T,s)\,x(s)| \, ds + \int_t^T |C(t_0) - C(t)| \; |[A + B\,C(T)]^{-1} B| \; |R(T,s)\,x(s)| \, ds.$$

Letting

$$k_1 = |[A + B\,C(T)]^{-1} B|,$$

and applying the Holder inequality to the first term in the right hand side of the above inequality, we obtain

$$|(\tilde{L}x)(t) - (\tilde{L}x)(t_0)| \leq \left(\int_0^t |R(t,s) - R(t_0, s)|^q \, ds \right)^{\frac{1}{q}} \left(\int_0^t |x(s)|^p \, ds \right)^{\frac{1}{p}} + k_1 \int_0^T |C(t_0) - C(t)| \; |R(T,s)\,x(s)| \, ds.$$

Since $C(t)$ is absolutely continuous on $[0, T]$, and using the other property that we obtained in [8] for $R(t, s)$, namely

$$\int_0^t | R(t,s) - R(t_0, s) |^q \, ds \to 0 \qquad \text{as} \quad t \to t_0 ,$$

the set

$$\left\{ (\tilde{L}x)(t) : \ x \in L^p([0, T], \mathbb{R}^n) , \ |x|_{L^p} \leq M_2 , M_2 > 0 \right\}$$

is equicontinuous on $[0, T]$. Therefore, by Arzela - Ascoli theorem, $\tilde{L}$ is a compact operator.

Theorem 2 Consider equation (2.3) and assume the following conditions are satisfied:

(i) the operator $\tilde{L}$ defined by (2.6) is a compact operator from $L^q([0, T], \mathbb{R}^n)$ into $L^p([0, T], \mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$(x , \tilde{L}x) = \int_0^T \int_0^T \tilde{R}(t,s) \, x(s) \, x(t) \, dt \, ds \geq 0.$$

holds true for any $x \in L^q([0, T], \mathbb{R}^n)$, $1< q < \infty$.

(ii) $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Caratheodory condition (f is measurable in (t, x) and for each fixed $t \in [0, T]$, f is continuous in x); and, the Niemytski operator F given by (2.5) takes $L^p([0, T], \mathbb{R}^n)$ into $L^q([0, T], \mathbb{R}^n)$.

(iii) there exist $\psi \in L^q([0, T], \mathbb{R}^n)$, and constants $\alpha > 0$, $c > 0$ such that

$$(f(t,x) - \psi(t)) \cdot x \geq c \, | f(t,x) - \psi(t) | \cdot |x| \qquad \text{for} \quad |x| \geq \alpha \quad \text{and} \quad t \in [0, T].$$

Then equation (2.3) has a solution.

The proof of theorem 2 can be found in [6], which has been conducted following the same approach as in the paper by H. Brezis and F. Browder [4].

The above theorem requires that $\tilde{L}$ be a compact operator, which is a severe restriction. Removing the compactness condition would lead to a substantial improvement of the existence result. This is achieved by applying the theory of monotone operators. Results regarding this case can be found in [6]. There are also results available using monotone operators in non-reflexive spaces : for instance, see [3].

Remark 2: If we consider the Sturm-Liouville problem

$$\dot{x}(t) = (Lx)(t) + \lambda\, h(t)\, x(t)$$

$$A\, x(0) + B\, x(T) = \theta$$

where $h \in L^2([0, T], \mathbb{R}^n)$, then it can be reduced to

$$x(t) = \lambda \int_0^T \tilde{R}(t,s)\, h(s)\, x(s)\, ds,$$

and letting $R_1(t,s) = \tilde{R}(t,s)\, h(s)$, we obtain

$$x(t) = \lambda \int_0^T R_1(t,s)\, x(s)\, ds\ .$$

Therefore, the investigation of the Sturm-Liouville problem formulated above can be reduced to the classical Fredholm integral equation, of the second kind.

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