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SOME SIMPLE CRITERIA FOR STARLIKENESS AND CONVEXITY

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ABSTRACT. Let A_n denote the class of functions $f(z) = z + a_{n+1} z^{n+1} + \dots$, $n \geq 1$, that are analytic on the unit disc U and let $A = A_1$. Let us denote by S^* and K the subclasses of A consisting of functions which are starlike and convex, respectively.

We find some criteria involving f' or f'' only in determining the starlikeness or convexity of f in A_n or of $F = I(f)$, where I is a certain integral operator. Several examples will show how difficult it is to use the analytic definitions of starlikeness or convexity, while our criteria produce results immediately.

1. INTRODUCTION.

Let A_n denote the class of functions

$$f(z) = z + a_{n+1} z^{n+1} + \dots, \quad n \geq 1,$$

that are analytic in the unit disc $U = \{z \in \mathbb{C};\ |z| < 1\}$ and let $A = A_1$.

The analytic function f, with $f(0) = 0$ and $f'(0) \neq 0$, is starlike on U (i.e. f is univalent in U and $f(U)$ is starlike with respect to the origin) iff $\mathrm{Re}[zf'(z)/f(z)] > 0$, for $z \in U$. The analytic function f, with $f'(0) \neq 0$, is convex in U (i.e. f is univalent in U and $f(U)$ is convex) iff $\mathrm{Re}[zf''(z)/f'(z)] + 1 > 0$, for $z \in U$. Let denote by S^* and K the subclasses of A consisting of functions f which are starlike and convex, respectively.

Key words: starlike functions; convex functions; integral operators.

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In several recent papers, [1], [3], [5-10], some simple sufficient conditions for starlikeness or convexity were given. In this paper we obtain other new such criteria for starlikeness or convexity.

2. PRELIMINARIES.

Let F and G be two analytic functions in U. If G is univalent, then we say that F is subordinate to G, written $F \prec G$, or $F(z) \prec G(z)$, iff $F(0) = G(0)$ and $F(U) \subset G(U)$.

We shall use the following lemmas in order to prove our main results.

LEMMA 1. [2] *Let* $c > -1$ *and let* q *be convex in* U, *with* $q(0) = 1$ *and let* $P(z) = 1 + P_n z^n + \ldots$ *be analytic in* U. *If*

$$P(z) + \frac{1}{c+1} z P'(z) \prec q(z),$$

then $P \prec Q$, *where*

$$Q(z) = \frac{c+1}{n z^{(c+1)/n}} \int_0^z q(t)\, t^{(c+1)/n-1}\, dt.$$

LEMMA 2. *Let* n *be a positive integer and let* $\beta_0 = \beta_0(n)$ *be the solution of the equation* $\beta\pi = 3\pi/2 - \arctan(n\beta)$. *Let*

$$\alpha = \alpha(\beta,n) = \beta + \frac{2}{\pi} \arctan(n\beta),$$

for $0 < \beta \le \beta_0$. *If* $p(z) = 1 + p_n z^n + \ldots$ *is analytic in* U, *then*

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \Rightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

This lemma was proved in [4, Theorem 5] in the case $n = 1$.

3. Main Results.

Theorem 1. *Let* $c > -1$, *let* n *be a positive integer and let us define*

$$M = M_n(c) = \frac{n+c+1}{(c+1)\left[\sqrt{(n+c+1)^2+(c+1)^2} + |c|\right]}. \tag{1}$$

If $f \in A_n$ *and*

$$|f'(z) - 1| < M, \tag{2}$$

then $I_c(f) \in K$, *where* $I_c\colon A_n \to A_n$ *is the integral operator defined by* $F = I_c(f)$, *with*

$$F(z) = \frac{c+1}{z^c} \int_0^z f(t)t^{c-1}\,dt = (c+1)\int_0^1 f(tz)\,t^{c-1}\,dt, \quad z \in U. \tag{3}$$

Proof: Let $f \in A_n$ satisfy (2), with M given by (1). From (3) we deduce

$$zF' + cF = (c+1)f$$

and

$$zF'' + (c+1)F' = (c+1)f'. \tag{4}$$

If we set $P = F'$, then (2) is equivalent to

$$P(z) + \frac{1}{c+1}zP'(z) = f'(z) \prec 1 + Mz \equiv q(z)$$

and by Lemma 1 we deduce $P \prec Q$, where

$$Q(z) = \frac{c+1}{nz^{(c+1)/n}} \int_0^z q(t)\, t^{(c+1)/n-1}\, dt = 1 + \frac{(c+1)M}{c+1+n} z.$$

If we let

(5) $$R = \frac{(c+1)M}{c+1+n},$$

then

(6) $$|P(z) - 1| < R.$$

Since $R < 1$, we deduce $|F' - 1| < 1$, which shows that F is univalent. Therefore, if we put

$$p(z) = \frac{zF''(z)}{F'(z)} + 1,$$

then the function $p(z) = 1 + p_n z^n + \ldots$ is analytic in U and from (4) we deduce

$$F'(p + c) = (c + 1)f'.$$

Hence (2) becomes

(7) $$|P(z)[p(z) + c] - (c + 1)| < (c + 1)M.$$

If $\mathrm{Re}\, p(z) \not> 0$, then there exists $z_0 \in U$ such that $\mathrm{Re}\, p(z_0) = is$. Therefore, in order to show that (7) implies $\mathrm{Re}\, p(z) > 0$ it is sufficient to check the inequality

(8) $$|P(z)[is + c] - (c + 1)| \geq (c + 1)M,$$

for all real s and all $z \in U$.

If we let $P = u + iv$, then

$$\begin{aligned} E &= |P(is + c) - (c + 1)|^2 \\ &= (u^2 + v^2)s^2 + 2v(c + 1)s + [cu - (c + 1)]^2 + c^2v^2 \\ &= (u^2 + v^2)s^2 + 2v(c + 1)s + |cP - (c + 1)|^2. \end{aligned}$$

On the other hand, from (6) we deduce

$$|cP - (c + 1)| \geq 1 - |c|R,$$

hence,

$$E \geq (u^2 + v^2)s^2 + 2v(c + 1)s + (1 - |c|R)^2$$

and the inequality (8) holds if

$$E - (c + 1)^2M^2 \geq (u^2 + v^2)s^2 + 2v(c + 1) + (1 - |c|R)^2 - (c + 1 + n)^2R^2 \geq 0$$

and this last inequality holds if

$$\Delta = (c + 1)^2v^2 - (u^2 + v^2)[(1 - |c|R)^2 - (c + 1 + n)^2R^2] \leq 0,$$

i.e. if

(9) $$v^2[(c + 1)^2 + (c + 1 + n)^2R^2 - (1 - |c|R)^2)] \leq u^2[(1 - |c|R)^2 - (c + 1 + n)^2R^2].$$

From (6) we deduce

$$\frac{v^2}{u^2} \leq \frac{R^2}{1 - R^2}$$

and a simple calculation yields

$$\frac{R^2}{1 - R^2} \leq \frac{(1 - |c|R)^2 - (c + 1 + n)^2R^2}{(c + 1)^2 + (c + 1 + n)^2R^2 - (1 - |c|R)^2},$$

where R is given by (5).

Hence the inequality (9) holds and from (8) and (7) we deduce $\operatorname{Re} p(z) > 0$, which shows that $F \in K$. ■

COROLLARY 1.1. *Let* $c > -1$ *and let* n *be a positive integer. If* $f \in A_n$ *and*

$$|f''(z)| \leq nM, \text{ for } z \in U,$$

where $M = M_n(c)$ *is given by (1), then* $I_c(f) \in K$, *where* I_c *is defined by (3).*

If in Theorem 1 we take $c = 0$ and use the fact that $F \in K \Leftrightarrow f \in S^*$, then we deduce the following criteria for starlikeness.

COROLLARY 1.2. *If* $f \in A_n$ *and*

$$|f'(z) - 1| < \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \text{ for } z \in U,$$

then $f \in S^*$.

We note that for $n = 1$ this corollary was obtained in [11, Theorem 3]. Another proof was given in [6].

COROLLARY 1.3. *If* $f \in A_n$ *and*

$$|f''(z)| \leq \frac{n(n+1)}{\sqrt{(n+1)^2 + 1}}, \text{ for } z \in U,$$

then $f \in S^*$.

This corollary improves a result obtained in [8, Theorem 2].

COROLLARY 1.4. *Let* $c > -1$ *and let* n *be a positive integer. If* $f \in A_n$ *and*

(10) $$|f'(z)-1| \le \frac{(n+1)(c+1+n)}{(c+1)\sqrt{(n+1)^2+1}}, \text{ for } z \in U,$$

then $I_c(f) \in S^*$, *where* I_c *is defined by (3).*

Proof: From (4) and (10) we deduce

$$F' + \frac{1}{c+1}\, zF'' = f' \prec 1 + \frac{(n+1)(c+1+n)}{(c+1)\sqrt{(n+1)^2+1}}\, z$$

and by Lemma 1 we obtain

$$F'(z) \prec 1 + \frac{n+1}{\sqrt{(n+1)^2+1}}\, z.$$

Hence

$$|F'(z)-1| < \frac{n+1}{\sqrt{(n+1)^2+1}}$$

and from Corollary 1.2 we deduce $F \in S^*$.

THEOREM 2. *If* $f \in A_n$ *and*

$$|f''(z)| \le \frac{n}{n+1}, \text{ for } z \in U$$

then $|f''(z)/f'(z)| \le 1$ *and hence* $f \in K$. *This result is sharp.*

Proof: By Schwarz's lemma we have

$$|f''(z)| \le \frac{n}{n+1}\, |z|^{n-1}.$$

Hence

$$\left|\int_0^z f''(\zeta)d\zeta\right| = \left|z\int_0^1 f''(tz)dt\right| \le \frac{n}{n+1}\int_0^1 t^{n-1}dt = \frac{1}{n+1}.$$

Since

$$f'(z) = 1 + \int_0^z f''(\zeta)d\zeta,$$

we deduce

$$|f'(z)| \geq 1 - |\int_0^z f''(\zeta)d\zeta| \geq 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Therefore $|f''(z)/f'(z)| \leq 1$, which shows that $f \in K$. The function $f(z) = z + \frac{z^{n+1}}{(n+1)^2}$ shows that this result is sharp. ■

THEOREM 3. *If* $f \in A_n$ *and*

(11) $$|\arg f'(z)| < \phi_n = \alpha_n \frac{\pi}{2},$$

where $\alpha = \alpha_n$ *is the unique root of the equation*

$$2 \arctan[n(1-\alpha)] + \pi(1-2\alpha) = 0,$$

then $f \in S^*$.

Proof: If we let $p(z) = zf'(z)/f(z)$ and $P(z) = f(z)/z$, then by Lemma 2 the inequality (11) implies

$$|\arg P(z)| < \beta_n \frac{\pi}{2},$$

where $\beta_n = 1 - \alpha_n$ and we deduce

$$|\arg p(z)| = \left|\arg \frac{f'(z)}{P(z)}\right| \leq |\arg f'(z)| + |\arg P(z)| < (\alpha_n + \beta_n)\frac{\pi}{2} = \frac{\pi}{2}$$

for $z \in U$, which shows that $f \in S^*$. ■

We note that

$$\alpha_1 = 0.6165\ldots, \qquad \phi_1 = 0.968\ldots \quad (55°,48\ldots)$$
$$\alpha_2 = 0.6808\ldots, \qquad \phi_2 = 1.068\ldots \quad (61°,23\ldots).$$

In the case $n = 1$, Theorem 4 was proved in [6, Theorem 1].

4. EXAMPLES.

EXAMPLE 1. If we let

$$f(z) = (1-\lambda)z + \lambda \sin z,\ z \in U,\ \lambda \in \mathbb{C}, \tag{12}$$

then we have

$$|f'(z) - 1| = 2\left|\lambda \sin^2 \frac{z}{2}\right| < 2|\lambda| \sinh^2(1/2) = |\lambda| \frac{(e-1)^2}{2e}.$$

Since $f \in A_2$, from Corollary 1.2, with $n = 2$, we deduce that

$$|\lambda| \le \frac{6e}{(e-1)^2\sqrt{10}} = 1.746\ldots \Rightarrow f \in S^*.$$

We also have

$$|f''(z)| = |\lambda \sin z| < |\lambda| \sinh 1$$

and from Theorem 3, with $n = 2$, we deduce that

$$|\lambda| \le \frac{2}{3\sinh 1} = \frac{4e}{3(e^2-1)} = 0.567\ldots \Rightarrow f \in K.$$

In particular the function $f(z) = z + \sin z$ is convex in U.

EXAMPLE 2. Let $f(z) = z + \lambda[e^z - 1 - z - z^2/2]$, $z \in U$, $\lambda \in \mathbb{C}$. In this case we have

$$|f'(z) - 1| < |\lambda|(e - 2).$$

Since $f \in A_2$, from Corollary 1.2, with $n = 2$, we deduce that

$$|\lambda| \le \frac{3}{(e-2)\sqrt{10}} = 1.32... \Rightarrow f \in S^*.$$

We also have

$$|f''(z)| = |\lambda(e^z - 1)| < |\lambda|(e - 1)$$

and from Theorem 2, with $n = 2$, we deduce that

$$|\lambda| \le \frac{2}{3(e-1)} = 0.387... \Rightarrow f \in K.$$

EXAMPLE 3. Consider the integral operator (3) with $c = 1$, i.e.

$$F(z) = \frac{2}{z}\int_0^z f(t)dt.$$

If f is given by (12), then

$$F(z) = (1 - \lambda)z + \frac{2\lambda}{z}(1 - \cos z)$$

and applying Corollary 1.4, with $c = 1$ and $n = 2$, we deduce that

$$|\lambda| \le \frac{12e}{(e-1)^2\sqrt{10}} = 3.493... \Rightarrow F \in S^*.$$

EXAMPLE 4. If we let

$$f(z) = z + \lambda \int_0^z \int_0^t \frac{s}{e^s - 1}\, ds dt,\ z \in U,\ \lambda \in \mathbb{C},$$

then

$$|f''(z)| < |\lambda| \frac{e}{e-1}$$

and from Corollary 1.3, with $n = 1$, we deduce that

$$|\lambda| \leq \frac{2(e-1)}{e\sqrt{5}} = 0.565... \Rightarrow f \in S^*.$$

Also, according to Theorem 2, we obtain

$$|\lambda| \leq \frac{e-1}{2e} = 0.316... \Rightarrow f \in K.$$

EXAMPLE 5. If we let

$$f(z) = z + \lambda \int_0^z \int_0^t \left(\frac{s}{e^s - 1} - 1\right) ds dt,\ z \in U,\ \lambda \in \mathbb{C},$$

then $f \in A_2$ and

$$f''(z) = \lambda \left(\frac{z}{e^z - 1} - 1\right).$$

Since for $|z| = r < 2\pi$ we have

$$\left|\frac{z}{e^z - 1} - 1\right| < 1 + \frac{r}{2}\left(1 - \cot\frac{r}{2}\right),$$

we deduce

$$|f''(z)| < \frac{|\lambda|}{2}\left(3 - \cot\frac{1}{2}\right).$$

Hence by Corollary 1.3, with $n = 2$, we deduce

$$|\lambda| \le \frac{12}{\left(3 - \cot\frac{1}{2}\right)\sqrt{10}} = 3.244... \Rightarrow f \in S^*$$

and by Theorem 2 we obtain

$$|\lambda| \le \frac{4}{3\left(3 - \cot\frac{1}{2}\right)} = 1.14... \Rightarrow f \in K.$$

EXAMPLE 6. Consider the integral operator (3), with $c = -1/2$, i.e.

$$F(z) = \frac{1}{2}\int_0^1 f(tz)\, t^{-3/2}\, dt.$$

If f is defined by (12), then

$$F(z) = (1 - \lambda)z + \frac{\lambda}{2}\int_0^1 t^{-3/2}\sin(tz)dt.$$

Applying Theorem 1, with $c = -1/2$ and $n = 2$, we deduce that

$$|\lambda| \le \frac{20e}{(e-1)^2(\sqrt{26}+1)} = 3.01... \Rightarrow F \in K$$

and from Corollary 1.3, with $n = 2$, we obtain

$$|\lambda| \le \frac{30e}{(e-1)^2\sqrt{10}} = 8.73\ldots \Rightarrow F \in S^*.$$

EXAMPLE 7. If we let

$$f(z) = \int_0^z \left(\frac{1+t^2}{1-t^2}\right)^{2/3} dt,\ z \in U,$$

then $f \in A_2$ and

$$|\arg f'(z)| < \frac{\pi}{3}.$$

Hence by Theorem 3, with $n = 2$, we deduce that $f \in S^*$.

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