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RINGS WITH AN ALMOST DIVISION ALGORITHM

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Abstract.

The concept and some algebraic properties of additively absorptive subrings of order $t \geq 1$ (t a fixed integer) in a Euclidean ring (domain) is discussed. For a fixed integer $t \geq 1$, a unitary proper subring D of a Euclidean ring R with Euclidean function φ is said to be (additively) absorptive of order t in R, if for each $f \in R \backslash D$ there exists $g \in R$ such that $f+g \in D$ and $1 \leq \varphi(g) \leq t$. The main results of the paper are as follows: If D is an absorptive ring of order t in a Euclidean domain R with a Euclidean function φ which satisfies the properties that, for all $f, g, r \in R$, $\varphi(fg) = \varphi(f) + \varphi(g)$ and $\varphi(r) < \varphi(g)$ implies $\varphi(r+g) = \varphi(g)$, then for each $f, g \in D$, $g \neq 0$, there exist $q, r \in D$ such that $f = qg + r$ with $r = 0$ or $\varphi(r) < \varphi(g)$ or $\varphi(r) = i + \varphi(g)$ for some $1 \leq i \leq t$. Furthermore, it is shown that if $I \neq 0$ is an ideal of D and $\varphi(I) = \inf\{\varphi(f) \mid f \in I\}$, then I can be generated by $(t+1)$ or fewer elements. In addition, if for each $a \in R$, $1 \leq \varphi(a) \leq t$ implies $a \notin D$, and I contains an element of φ value equal to $i + \varphi(I)$ for some $1 \leq i \leq t$, then I cannot be a principal ideal in D. Examples of both absorptive and non-absorptive subrings of $k[x]$ (the ring of polynomials over a field k) are given.

Introduction.

This paper is basically motivated by the work of Nick H. Vaughan [3]. In [3] it is shown that the subring $D = \{f \in k[x] \mid x \text{ coefficient of } f \text{ is zero}\}$ of $k[x]$, k a field, satisfies the following properties:

1) For any f, $g \in D$, $g \neq 0$, there exist q, $r \in D$ such that $f = qg + r$ with $r = 0$, or $\deg r < \deg g$, or $\deg r = 1 + \deg g$.

2) Any ideal I of D can be generated by one or two elements.

In [1, Propositions 3 and 4], results 1 and 2 (above) are generalized as follows: For any fixed integer $t \geq 1$, let $D^{(t)} = \{f \in k[x] \mid x, x^2, \ldots, x^t$ coefficients of f are zero$\}$.

1′) For any $f, g \in D^{(t)}$, $g \neq 0$, there exist q, $r \in D^{(t)}$ such that $f = qg + r$ with $r = 0$, or $\deg r < \deg g$, or $\deg r = i + \deg g$ for some $1 \leq i \leq t$.

2′) Any ideal I in $D^{(t)}$ can be generated by $(t+1)$ or less elements.

The main purpose of this paper is a natural extension of Propositions 3 and 4 in [1] for some subrings of a special class of Euclidean rings. A unitary subring D of a unitary ring R is a subring of R with $1_D = 1_R$. If R is a Euclidean ring with a Euclidean function φ, it is well known that for all $a \in R \backslash \{0\}$, $\varphi(a) \geq \varphi(1)$, $\varphi(a) = \varphi(-a)$, and $\varphi(a) = \varphi(1)$ if and only if a is a unit in R.

In this note all rings are commutative, Z is the ring of rational integers, Q is the field of rational numbers, and $N = \{0, 1, 2, \ldots\}$ is the set of natural numbers.

1. Absorptive Subrings.

Definition. Let R be a Euclidean ring with Euclidean function φ, and assume $t \geq 1$ is a fixed integer. A unitary proper subring D of R is said to be additively absorptive (or simply, absorptive) of order t in R, if for each $f \in R \backslash D$ there exists $g \in R$ such that $f + g \in D$ and $1 \leq \varphi(g) \leq t$.

Example 1. $D^{(t)}$ as defined above is an example of an absorptive subring of order t in $k[x]$.

Example 2. Let $k[x]$ be ring of polynomials over a field k and assume $t \geq 1$ is a fixed odd integer. Define $R^{(t)} = \{f \in k[x] \mid x^j$ coefficient of f is zero for all odd j, $1 \leq j \leq t\}$. This is again an example of an absorptive subring of order t in $k[x]$.

Example 3. $Z[x]$ is not absorptive of order t in $Q[x]$ for any fixed integer $t \geq 1$. Let $f = 1 + x + x^2 + \ldots + x^t + \frac{1}{3}x^{t+1}$. It is clear that $f \notin Z[x]$. Now it is impossible to have a $g \in Q[x]$ with $1 \leq \deg g \leq t$ and $f + g \in Z[x]$.

Theorem 1. Let R be a Euclidean ring with Euclidean function φ, and let $t \geq 1$ be a fixed integer. Assume D_1 and D_2 are unitary proper subrings of R with $D_1 \subseteq D_2$. If D_2 is not absorptive of order t in R, then D_1 is not absorptive of order t in R.

Proof. Assume D_1 is absorptive of order t in R. Let $f \in R \backslash D_2$. Now by assumption there exists $g \in R$ such that $f+g \in D_1$ and $1 \leq \varphi(g) \leq t$. Hence we can conclude that D_2 is absorptive of order t in R, which is a contradiction.□

Remark. From Example 3 and Theorem 1 we can see that no unitary proper subring of $Z[x]$ can be absorptive of order t in $Q[x]$ for any integer $t \geq 1$.

2. Main Results.

Theorem 2. Let R be a Euclidean domain with Euclidean function φ with the following properties:

1) $\varphi(fg) = \varphi(f) + \varphi(g)$ for all $f,\ g \in R \backslash \{0\}$.

2) For all $r, g \in R$ if $\varphi(r) < \varphi(g)$, then $\varphi(r+g) = \varphi(g)$.

Assume $t \geq 1$ is a fixed integer and D is an absortive subring of order t in R. Then for any $f,\ g \in D,\ g \neq 0$, there exist $q,\ r \in D$ such that $f = qg + r$ with $r = 0$ or $\varphi(r) < \varphi(g)$ or $\varphi(r) = i + \varphi(g)$ for some $1 \leq i \leq t$.

Proof. Let $f, g \in D$ with $g \neq 0$. Since R is Euclidean, then there exist $q,\ r \in R$ such that $f = qg + r$ with $r = 0$ or $\varphi(r) < \varphi(g)$. If $q \in D$, then $r = f - qg \in D$ and we are done. Now suppose $q \notin D$, then by hypothesis there exists $\tilde{q} \in R$ such that $q + \tilde{q} \in D$ and $1 \leq \varphi(\tilde{q}) \leq t$. Since $f = (q+\tilde{q})g + r - \tilde{q}g$ and since $r - \tilde{q}g \in D$, it remains only to show that $1 + \varphi(g) \leq \varphi(r - \tilde{q}g) \leq t + \varphi(g)$. Clearly, $1 \leq \varphi(\tilde{q}) \leq t$ implies $1 + \varphi(g) \leq \varphi(\tilde{q}) + \varphi(g) \leq t + \varphi(g)$ and from condition (1) we obtain $1 + \varphi(g) \leq \varphi(\tilde{q}g) \leq t + \varphi(g)$. Since R is a Euclidean ring, we have $\varphi(-\tilde{q}g) = \varphi(\tilde{q}g)$. Thus $1 + \varphi(g) \leq \varphi(-\tilde{q}g) \leq t + \varphi(g)$, and by applying condition (2), it follows that $1 + \varphi(g) \leq \varphi(r - \tilde{q}g) \leq t + \varphi(g)$.□

For each ideal I of D, let $\varphi(I)$ be the smallest element of the set $\{\varphi(f) \mid \text{for all } f \in I\}$.

Thereom 3. Let R, φ, D, and t be as in Theorem 2 and let I be a nonzero ideal of D with $\varphi(I) = j$. Suppose that if $a \in R$ with $1 \leq \varphi(a) \leq t$ then necessarily $a \notin D$. Assume further that there exists $h \in I$ such that $\varphi(h) = i + j$

for some $1 \le i \le t$. Then

(i) I can be generated by $t+1$ or fewer elements;

(ii) I is not a principal ideal of D.

Remark. Suppose that the conditions of Theorem 3 are satisfied and that for $a \in R \backslash D$, $\varphi(a)=0$. Since D is absorptive, there exists $b \in R$ such that $a+b \in D$ with $1 \le \varphi(b) \le t$. Thus, $0=\varphi(a)<\varphi(b)$ and by Theorem 2 we have $1 \le \varphi(b)=\varphi(a+b) \le t$. By the hypotheses of Theorem 3 we must have $a+b \notin D$, a contradiction. We therefore must conclude that if $\varphi(a)=0$, then $a \in D$. We shall use this fact in the following proof of Theorem 3.

Proof. Choose an element $g \in I$ with $\varphi(g)=j$. By Theorem 2, for each $f \in I$, there exist q, $r \in D$ such that $f=qg+r$ with (i) $r=0$ or (ii) $\varphi(r)<\varphi(g)$ or (iii) $\varphi(r)=i+\varphi(g)$ for some $1 \le i \le t$.

Now by the minimality of $\varphi(g)$ and the fact that $r=f-qg \in I$, it is obvious that $\varphi(r)<\varphi(g)$ cannot happen. And for those $f \in I$ such that $r=0$, it is clear that $f \in (g)$. Indeed, if for each $f \in I$ case (i) occurs, then we can conclude that $(g)=I$.

Next, suppose for some element $f \in I$, case (iii) occurs. This implies the existence of an element of I (namely $r \in I$) with φ value equal to $i+\varphi(g)$ for some $1 \le i \le t$. Now since $r \in I$ and $j<\varphi(r) \le t+j$, it is clear that the following set $C=\{\alpha \in N \mid I$ contains elements of φ value equal to α with $j<\alpha \le t+j\}$ is nonempty. Assume that the cardinality of C, $|C|=k$. Now, label the elements of C as α_1, α_2, ..., α_k, where $\alpha_1>\alpha_2>\ldots>\alpha_k$. By construction of C we can choose k elements f_{α_1}, f_{α_2}, ... ,f_{α_k} from I with $\varphi(f_{\alpha_i})=\alpha_i$, $i=1,2,\ldots,k$.

Now it is clear that $\varphi(r)=\varphi(f_{\alpha_{i_1}})$ for some $1 \le i_1 \le k$. Since R is a Euclidean ring, then there exist a_{i_1}, $r_1 \in R$ such that $r=a_{i_1}f_{\alpha_{i_1}}+r_1$ with $r_1=0$ or $\varphi(r_1)<\varphi(f_{\alpha_{i_1}})$. Assume that $r_1=0$. Then $r=a_{i_1}f_{\alpha_{i_1}}$ and $\varphi(r)=\varphi(a_{i_1})+\varphi(f_{\alpha_{i_1}})$ imply that $\varphi(a_{i_1})=0$, which by the above remark implies $a_{i_1} \in D$, and we obtain $f=qg+r=qg+a_{i_1}f_{\alpha_{i_1}} \in (g,f_{\alpha_{i_1}})$.

Now suppose $r_1 \ne 0$ and $\varphi(r_1)<\varphi(f_{\alpha_{i_1}})$. Thus, we have $\varphi(r_1)<\varphi(f_{\alpha_{i_1}}) \le$

$\varphi(a_{i_1})+\varphi(f_{\alpha_{i_1}})=\varphi(a_{i_1}f_{\alpha_{i_1}})$, which by hypothesis implies $\varphi(a_{i_1}f_{\alpha_{i_1}}+r_1)=\varphi(a_{i_1}f_{\alpha_{i_1}})$. Now we have $\varphi(r)=\varphi(a_{i_1}f_{\alpha_{i_1}}+r_1)=\varphi(a_{i_1}f_{\alpha_{i_1}})=\varphi(a_{i_1})+\varphi(f_{\alpha_{i_1}})$ which implies $\varphi(a_{i_1})=0$ and thus by the remark, $a_{i_1}\in D$. In this case, it is clear that $\varphi(r_1)=\varphi(f_{\alpha_{i_2}})$ for some $i_1<i_2\leq k$. Again by the division algorithm, there exist a_{i_2}, $r_2\in R$ such that $r_1=a_{i_2}f_{\alpha_{i_2}}+r_2$ with $r_2=0$ or $\varphi(r_2)<\varphi(f_{\alpha_{i_2}})$. By the same argument as we showed $a_{i_1}\in D$, it can be shown (in either case, $r_2=0$ or $\varphi(r_2)<\varphi(f_{\alpha_{i_2}})$) that $a_{i_2}\in D$, and whenever $r_2=0$, we have $f=qg+r=qg+a_{i_1}f_{\alpha_{i_1}}+r_1=qg+a_{i_1}f_{\alpha_{i_1}}+a_{i_2}f_{\alpha_{i_2}}\in(g,f_{\alpha_{i_1}},f_{\alpha_{i_2}})$.

Continuing the process as above, $r, r_1, r_2, \ldots$ are elements in I with $j+t\geq\varphi(r)>\varphi(r_1)>\varphi(r_2)>\ldots\geq j$. Thus we reach an element $r_s\in I$ with $\varphi(r_s)=\varphi(g)$ and $r_{s+1}=0$. Actually by the division algorithm in R, there exist q', $r_{s+1}\in R$ such that $r_s=q'g+r_{s+1}$ with $r_{s+1}=0$ or $\varphi(r_{s+1})<\varphi(g)$. The minimality of $\varphi(g)$ and the fact that $r_{s+1}=r_s-q'g\in I$ forces the impossibility of $\varphi(r_{s+1})<\varphi(g)$. Hence $r_s=q'g$ and $\varphi(r_s)=\varphi(q')+\varphi(g)$, which implies $\varphi(q')=0$, and by the preceding remark, $q'\in D$. Thus $f=qg+a_{i_1}f_{\alpha_{i_1}}+a_{i_2}f_{\alpha_{i_2}}+\ldots+q'g\in(g,f_{\alpha_1},f_{\alpha_2},\ldots,f_{\alpha_k})$, which proves that $I=(g,f_{\alpha_1},f_{\alpha_2},\ldots,f_{\alpha_k})$, where α_1, $\alpha_2,\ldots,\alpha_k\in C$.

Finally assume that for each $a\in R$, $1\leq\varphi(a)\leq t$ implies $a\notin D$, and I contains an element h with $\varphi(h)=i+\varphi(g)$ for some $1\leq i\leq t$. In this case r cannot be zero for all $f\in I$. Suppose the contrary, then $h=qg$ for some $q\in D$. Thus, $\varphi(h)=i+j=\varphi(q)+j$ implies $1\leq\varphi(q)=i\leq t$, and by assumption this makes $q\notin D$, which is a contradiction. From this we can conclude $h\notin(g)$, i.e., $(I)\not\subseteq(g)$. In other words, no element of I with φ value equal to j can generate I in D. Now, suppose there exists $g'\in I$ which generates I in D. From the above argument and minimality of $j=\varphi(g)$, we must have $\varphi(g')>\varphi(g)$. Since $g\in I$, $g=q'g'$ for some $q'\in D$ and $\varphi(g)=\varphi(q')+\varphi(g')\geq\varphi(g')$, which is a contradiction.□

In view of the above theorem, clearly D is a Noetherian domain.

3. Some Properties of $D^{(t)}$.

For the definition of $D^{(t)}$ we refer the reader to the introduction of this paper. In this section we mention some properties of $D^{(t)}$ as a general form of $D^{(1)}$ which was studied in [3, Sec. 3].

It is fairly routine to show that $D^{(t)}$ satisfies the properties of $D(D^{(1)})$ stated in [3, Sec. 3]. Additionally, since $D^{(t)}$ is not integrally closed, $D^{(t)}$ is clearly neither Prufer nor Dedekind. $D^{(t)}$ is furthermore not a valuation domain [2, pp. 12-14], nor a pseudo-Bezout domain [2, p. 15] (a domain is pseudo-Bezout if every pair of elements has a greatest common divisor). Thus, $D^{(t)}$ is also not Bezout.

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