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ON A NUMERICAL ALGORITHM FOR APPROXIMATING THE SOLUTION IN THE THEORY OF MINDLIN PLATES

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The theory of plates with transverse shear deformation [1] provides more complete information on the behaviour of an elastic plate than Kirchhoff's classical theory [2]. However, the system of partial differential equations describing it is more complicated than the biharmonic equation and requires careful handling. This is particularly relevant when a numerical approximation of the solution is desired, because the operators of the associated boundary integral equations are singular. In what follows we study numerically an algorithm for the construction of an approximate solution and identify, in the specific case of a circular plate, the range of two of the parameters where the method seems to produce optimum results.

In a finite domain S^+ bounded by a closed C^2-curve ∂S in $\mathbf{R}^2$ we consider the system of partial differential equations [3]

$$A(\partial_x)u(x) = 0, \tag{1}$$

where $A(\partial_x) = A(\partial/\partial x_1, \partial/\partial x_2)$, $A(\xi_1, \xi_2)$ is the matrix

$$\begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix},$$

$x = (x_1, x_2)$ is a generic point in $\mathbf{R}^2$, $u = (u_1, u_2, u_3)^{\mathrm{T}}$ is a vector characterizing the displacements, λ and μ are the Lamé constants of the (homogeneous and isotropic) material, $h^2 = h_0^2/12$, h_0 is the constant thickness of the plate, and $\Delta = \xi_1^2 + \xi_2^2$. The corresponding boundary stress operator can be written as $T(\partial_x)$, where $T(\xi_1, \xi_2)$ is

the matrix

$$\begin{pmatrix} h^2(\lambda+2\mu)\nu_1\xi_1 + h^2\mu\nu_2\xi_2 & h^2\mu\nu_2\xi_1 + h^2\lambda\nu_1\xi_2 & 0 \\ h^2\lambda\nu_2\xi_1 + h^2\mu\nu_1\xi_2 & h^2\mu\nu_1\xi_1 + h^2(\lambda+2\mu)\nu_2\xi_2 & 0 \\ \mu\nu_1 & \mu\nu_2 & \mu(\nu_1\xi_1 + \nu_2\xi_2) \end{pmatrix}$$

and $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the outward unit normal to ∂S.

We introduce the singular matrix solutions $D(x,y)$ and $P(x,y)$, where

$$\begin{aligned} D(x,y) &= (\operatorname{adj} A)(\partial_x)t(x,y), \\ P(x,y) &= \big(T(\partial_y)D(x,y)\big)^{\mathrm{T}}, \\ t(x,y) &= a\big[(4h^2 + |x-y|^2)\ln|x-y| + 4h^2K_0(h^{-1}|x-y|)\big], \end{aligned}$$

$a = \big[8\pi h^2\mu^2(\lambda+2\mu)\big]^{-1}$, $|x-y| = \big[(x_1-y_1)^2 + (x_2-y_2)^2\big]^{1/2}$, $\operatorname{adj} A$ is the adjoint of A and K_0 is the modified Bessel function of order zero.

If $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ is a solution of (1), then there holds the Somigliana formula

$$\alpha(x)u(x) = \int_{\partial S} \big[D(x,y)T(\partial_y)u(y) - P(x,y)u(y)\big]\, ds(y), \tag{2}$$

where

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in S^+, \\ \frac{1}{2} & \text{if } x \in \partial S, \\ 0 & \text{if } x \in S^- \end{cases}$$

and $S^- = \mathbf{R}^2 \setminus (S^+ \cup \partial S)$.

We note that the space of rigid displacements for (1) is spanned by the columns $f^{(i)}$, $i = 1,2,3$, of the matrix

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & -x_2 & 1 \end{pmatrix}.$$

Formula (2) can be used to expand the solutions of the Dirichlet and Neumann problems for (1) in generalized Fourier series. For simplicity, we consider the Dirichlet problem in S^+, that is, the system (1) with the boundary condition

$$u = g \text{ (given)} \quad \text{on} \quad \partial S. \tag{3}$$

Let ∂S_* be a smooth closed curve lying completely outside $\bar{S}^+$, and let $\{x^{(n)}\}_{n=1}^{\infty}$ be a sequence of points densely distributed on ∂S_*. If $\theta^{(jk)}$, $j = 1,2,3$, are the columns of the matrices $D(x, x^{(k)})$, $k = 1,2,\ldots$, then it can be shown that the set

$$\mathcal{S} = \{f^{(i)},\ \theta^{(jk)},\ i,j = 1,2,3,\ k = 1,2,\ldots\}$$

is linearly independent on ∂S and complete in $L^2(\partial S)$.

By (2), the (unique) regular solution $u(x)$ of (1), (3) satisfies

$$u(x) = \int_{\partial S} D(x,y)b(y)\,ds(y) - F(x), \quad x \in S^+, \tag{4}$$

$$F(x) = \int_{\partial S} D(x,y)b(y)\,ds(y), \quad x \in S^-, \tag{5}$$

where

$$F(x) = \int_{\partial S} P(x,y)a(y)\,ds(y), \quad x \in \mathbf{R}^2 \setminus \partial S, \tag{6}$$

$$b(y) = (Tu)(y), \quad y \in \partial S.$$

This means that if the unknown function $b(y)$ can be approximated from (5) in terms of the function $F(x)$ given by (6), then the solution $u(x)$ can also be approximated from (4).

From (5) it follows that

$$F(x^{(k)}) = \int_{\partial S} D(x^{(k)},y)b(y)\,ds(y);$$

in other words, since $D(x,y) = D^{\mathrm{T}}(y,x)$,

$$F_j(x^{(k)}) = \int_{\partial S} (\theta^{(jk)})^{\mathrm{T}}\,ds, \quad j = 1,2,3, \quad k = 1,2,\ldots$$

Let $\{\theta^{(m)}\}_{m=1}^{\infty}$ be the set obtained by arranging the elements of $\mathcal{S}$ in the order

$$f^{(1)}, f^{(2)}, f^{(3)}, \theta^{(11)}, \theta^{(21)}, \theta^{(31)}, \ldots, \theta^{(1k)}, \theta^{(2k)}, \theta^{(3k)}, \ldots$$

Following the Gram-Schmidt procedure, we construct an orthonormal set $\{\omega^{(n)}\}_{n=1}^{\infty}$, where

$$\omega^{(n)} = \sum_{m=1}^{n} k_{nm}\theta^{(m)}, \quad n = 1, 2, \ldots,$$

and k_{nm} are known numbers.

Consider the truncations

$$b^{(n)} = \sum_{r=1}^{n} p_r \omega^{(r)}, \quad n = 1, 2, \ldots,$$

of the generalized Fourier series for b. The coefficients p_r are given by

$$p_r = \int_{\partial S} (\omega^{(r)})^{\mathrm{T}} b \, ds = \sum_{m=1}^{r} k_{rm} \int_{\partial S} (\theta^{(m)})^{\mathrm{T}} b \, ds, \quad r = 1, 2, \ldots$$

Then the sequence $\{u^{(n)}\}_{n=1}^{\infty}$ of truncations of u, where, by (4),

$$u^{(n)}(x) = \int_{\partial S} D(x, y) b^{(n)}(y) \, ds(y) - F(x), \quad x \in S^{+},$$

converges uniformly to u on any closed subdomain S' of S^{+}. Since the columns of $D(x, y)$ satisfy (1), so do all the approximate solutions $u^{(n)}$.

However, the numerical implementation of the above scheme needs careful handling even in the case of simple domains, because the Gram-Schmidt process is ill-conditioned. Its replacement with an alternative equivalent construction helps to improve matters to some extent, but the drawback is not completely removed, for other problems may occur, especially in iterations of high order where a very large number of vectors have to be orthonormalized and the accumulation of round-off error could have a severe adverse effect on the result.

We illustrate these remarks in the case when ∂S is a circle with the centre at the origin and radius 2,

$$a_1 = 32 \cos \varphi + 8 \cos 3\varphi,$$
$$a_2 = -32 \sin \varphi + 8 \cos 3\varphi,$$
$$a_3 = 12 \cos \varphi,$$

φ is the polar angle, $\lambda = \mu = 1$, $h = 0.5$, and ∂S_* is a circle of radius $r > 2$ concentric with ∂S, on which the points $x^{(n)}$ correspond to the polar angles

$$0,\ \pi,\ \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi,\ \tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi,\ \tfrac{5}{4}\pi,\ \tfrac{7}{4}\pi,\ \ldots \tag{7}$$

(It is clear that the set of these points is densely distributed on ∂S_*.) Also, we use Simpson's rule with 128 strips to estimate the integrals.

The computational results are gathered in two tables. Table 1 gives the values

	$(0,0)$	$(0.05,0)$	$(1,\frac{1}{4}\pi)$	$(1.95,\frac{1}{2}\pi)$
	−1.012586	2.234103	16.131064	0.994313
$u^{(3)}$	1.012586	−1.213415	−16.131064	−54.349476
	3.178343	4.374561	13.004786	−25.171489
	0.864184	1.766402	12.432682	0.431947
$u^{(9)}$	−0.864184	−0.921341	−12.432682	−42.090746
	1.199042	2.113147	2.147578	−16.237291
	0.310032	0.943231	9.044736	0.003041
$u^{(15)}$	−0.310032	0.312479	−9.044736	−39.213637
	0.643475	0.848312	0.933147	−14.394537
	0.005451	0.312475	8.073234	0.000033
$u^{(27)}$	−0.005451	0.003837	−8.073234	−38.523275
	0.044647	0.023946	0.083179	−12.911630
	0.000000	0.200500	8.000000	0.000000
$u^{(51)}$	0.000000	0.000000	−8.000000	−37.443791
	0.000000	0.017494	0.000000	−12.785459
	0.000000	0.200500	8.000000	0.000000
u	0.000000	0.000000	−8.000000	−37.459500
	0.000000	0.017494	0.000000	−12.158494

Table 1.

	$(0,0)$	$(0.05,0)$	$(1,\frac{1}{4}\pi)$	$(1.95,\frac{1}{2}\pi)$
	2.301407	1.516234	3.213042	1.243157
$r = 2.01$	−2.301407	1.237641	−3.213042	−77.276991
	1.936642	0.931152	−2.177362	−19.274146
	0.000000	0.172841	7.494904	0.000012
$r = 2.1$	0.000000	0.000000	−7.494904	−38.390362
	0.000000	0.016174	0.000000	−13.468235
	0.000000	0.200500	8.000000	0.000000
$r = 3$	0.000000	0.000000	−8.000000	−37.443791
	0.000000	0.017494	0.000000	−12.785459
	0.000000	0.200500	7.999994	0.000011
$r = 10$	0.000000	0.000000	−7.999994	−37.673960
	0.000000	0.017494	0.000000	−13.188391
	0.000102	0.801637	7.233413	0.002123
$r = 17$	−0.000102	0.432104	−7.233413	−38.665870
	0.000034	1.403786	−0.000327	−13.870864
	3.413259	2.159013	2.132046	5.144236
$r = 18$	−3.413259	2.964154	−2.132046	−45.312875
	0.914012	4.310846	−1.755318	−17.059394
	0.000000	0.200500	8.000000	0.000000
u	0.000000	0.000000	−8.000000	−37.459500
	0.000000	0.017494	0.000000	−12.158494

Table 2.

of the iterations $u^{(3)}$ (when only the rigid displacements $f^{(i)}$, $i = 1,2,3$, have been used), and $u^{(9)}$, $u^{(15)}$, $u^{(27)}$, and $u^{(51)}$, which correspond to the choice of all the points $x^{(k)}$ in (7) generated by going completely round the circle ∂S_* once, twice, three times, and four times, respectively. These values, computed for $r = 3$, are shown against the exact values of the three components of the solution u at four

different points in S^+ given in terms of their polar coordinates: $(0,0)$, $(0.05,0)$, $(1,\frac{1}{4}\pi)$, and $(1.95,\frac{1}{2}\pi)$.

It is obvious from Table 1 that $u^{(51)}$ is the best approximation and that the error increases as we approach the boundary ∂S, where the integral operators associated with the scheme are singular. The next 'natural' iteration, $u^{(99)}$, has also been calculated, but its accuracy near the boundary is worse.

Table 2 shows the iteration $u^{(51)}$ at the same points as in Table 1, but computed for six different values of r. It is interesting to note that the accuracy does not improve when ∂S_* shrinks towards ∂S, as it does in the case of the Laplace equation. The explanation must again reside in the singular nature of the relevant operators. Also, there appears to be an 'optimal' range for r within which $u^{(51)}$ coincides with u to six decimal places. When r increases beyond 17, the results of the computation start losing accuracy rapidly, even at the centre of the circle.

The numerical data has also been used in graphical form. Figure 1 shows the exact curve $u_1 = u_1(x_1,0)$ for $x_1 \in [0,2]$ and the values of $u_1^{(51)}$ at $(0,0)$, $(0.05,0)$, $(0.1,0)$, ..., $(1.95,0)$, $(1.97,0)$, and $(1.99,0)$. Figure 2 shows the same for $u_3 = u_3(x_1,0)$ and $u_3^{(51)}$.

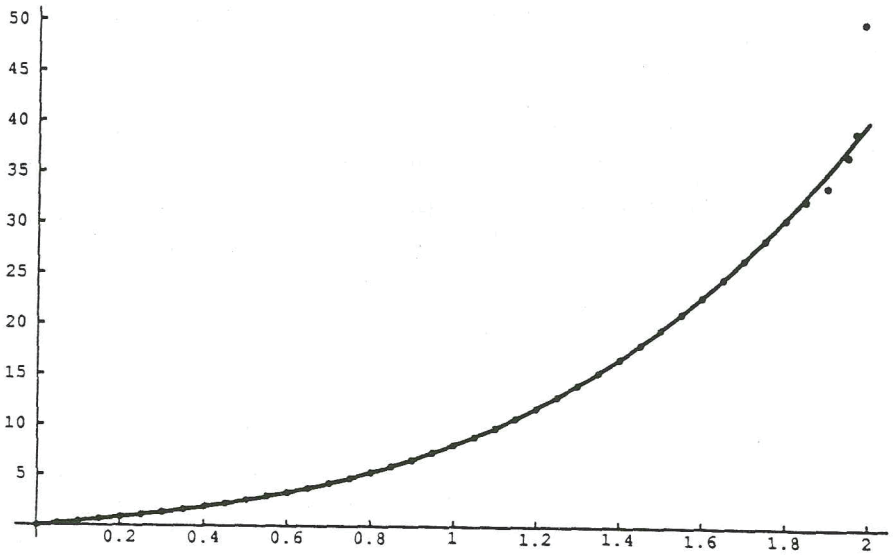

Figure 1.

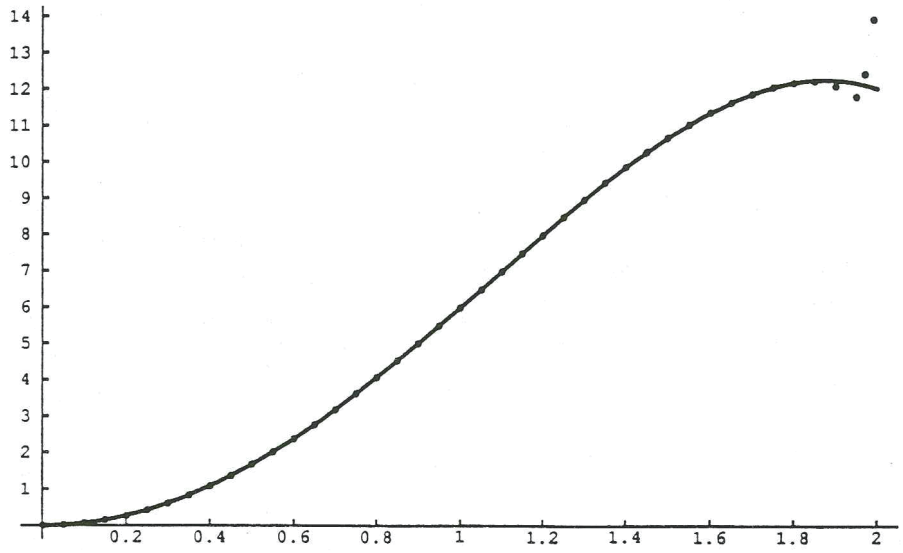

Figure 2.

References

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