

ON SOME NEW DISCRETE INEQUALITIES RELATED TO
A CERTAIN INTEGRAL INEQUALITY

B. G. Pachpatte

Abstract: In the present paper we establish some new discrete inequalities related to a certain integral inequality arising in the theory of differential equations. The inequalities obtained here can be used as handy tools in the theory of certain classes of finite difference equations.

1. Introduction

In [7] Lang Ou-Lang proved the following useful inequality.

LEMMA. Let u and f be real-valued non-negative continuous functions defined for all $t \geq 0$. If

$$(1) \quad u^2(t) \leq c^2 + 2 \int_0^t f(s)u(s)ds,$$

for all $t \geq 0$, where $c \geq 0$ is a constant, then

$$(2) \quad u(t) \leq c + \int_0^t f(s)ds,$$

for all $t \geq 0$.

The importance of this inequality stems from the fact that it is naturally applicable in the analysis of certain differential equations for which the other available inequalities do not apply directly. In the past few years many papers on integral and discrete inequalities centered around the well-known Gronwall inequality have appeared, but it seems that investigators have skipped to notice the significance of this inequality in the literature, see [1-3, 8-11]. In view of the important role played by the discrete inequalities in the study of numerical solutions of various discrete iterative models arising in science and technology, it is desirable to find some new discrete generalizations and extensions of this inequality which would be equally important in certain new applications. The main purpose of the present paper

is to establish some new discrete inequalities related to the above inequality which can be used as handy tools in the analysis of certain classes of finite difference equations. The inequalities that we propose here are motivated by the significant applications of the above inequality given in [4-7, 12] and believe that these inequalities will prove their importance to achieve a diversity of desired goals in various new applications.

2. Statement of Results

We first give some basic notations and definitions which will be used throughout this paper. Let $N_0 = \{0, 1, 2, \dots\}$, Δ is the operator defined by $\Delta u(t) = u(t+1) - u(t)$, $t \in N_0$ and for $i \geq 2$, $\Delta^i u(t) = \Delta(\Delta^{i-1} u(t))$, where $u(t)$ is a function defined for $t \in N_0$. For all $t_1 > t_2, t_1, t_2 \in N_0$ and any function $u(t)$ defined on N_0 , we use the usual conventions $\sum_{s=t_1}^{t_2} u(s) = 0$ and $\prod_{s=t_1}^{t_2} u(s) = 1$. The operators L_j are recursively defined by

$$L_0 u(t) = u(t), \quad L_j u(t) = \frac{1}{p_j(t)} \Delta L_{j-1} u(t), \quad j = 1, \dots, n,$$

with $p_n(t) = 1$, where $u(t)$ and $p_j(t) > 0$ are some functions defined on N_0 . For $t \in N_0$ and some functions $p(t)$ and $p_j(t) > 0, j = 1, \dots, n-1$, we set

$$A[t, p_1, \dots, p_{n-1}, p] \\ = \sum_{s_1=0}^{t-1} p_1(s_1) \dots \sum_{s_{n-1}=0}^{s_{n-2}-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p(s_n)$$

where $s_0 = t$.

For any real-valued function $v(x, y)$, $x, y \in N_0$, we define the operators $\Delta_1 v(x, y) = v(x+1, y) - v(x, y)$, $\Delta_2 v(x, y) = v(x, y+1) - v(x, y)$. We write $\Delta_1^n = \Delta_1 \times \dots \times \Delta_1$ (n times), $\Delta_2^m = \Delta_2 \times \dots \times \Delta_2$ (m times), and $\Delta_2^m \Delta_1^n v(x, y) = \Delta_2^m [\Delta_1^n v(x, y)]$. We often use the letters x and y to denote the two independent variables which are the members of N_0 . For $x, y \in N_0$ and some function $b(x, y)$ defined for $x, y \in N_0$, we set

$$B[x, y, b] \\ = \sum_{s_{n-1}=0}^{x-1} \sum_{s_{n-2}=0}^{s_{n-1}-1} \dots \sum_{s_0=0}^{s_1-1} \sum_{t_{m-1}=0}^{y-1} \sum_{t_{m-2}=0}^{t_{m-1}-1} \dots \sum_{t_0=0}^{t_1-1} b(s_0, t_0),$$

where $s_1 = x$, $t_1 = y$.

Let the product $N_0 \times \dots \times N_0$ (n times) be denoted by N_0^n . A point (x_1, \dots, x_n) in N_0^n is denoted by x . For any function $w(x)$ defined on N_0^n , we define the

operators $\Delta_1 w(x) = w(x_1 + 1, x_2, \dots, x_n) - w(x), \dots, \Delta_n w(x) = w(x_1, \dots, x_{n-1}, x_n + 1) - w(x)$. The operators I_j are recursively defined by

$$I_0 w(x) = w(x), I_j w(x) = \frac{1}{q_j(x)} \Delta_j I_{j-1} w(x), \quad j = 1, \dots, n,$$

with $q_n(x) = 1$, where $w(x)$ and $q_j(x) > 0$ are functions defined on N_0^n . For $x, y \in N_0^n$ and some functions $q(x)$ and $q_j(x) > 0, j = 1, \dots, n-1$, we set

$$M\{x_1, \dots, x_n, q_1, \dots, q_{n-1}, q\} \\ = \sum_{y_1=0}^{x_1-1} q_1(y_1, x_2, \dots, x_n) \dots \sum_{y_{n-1}=0}^{x_{n-1}-1} q_{n-1}(y_1, \dots, y_{n-1}, x_n) \sum_{y_n=0}^{x_n-1} q(y).$$

Our first result deals with a fairly general discrete version of the inequality given in Lemma.

THEOREM 1. Let $F(t) \geq 0, p(t) \geq 0, p_i(t) > 0$ for $i = 1, \dots, n-1$, be real valued functions defined for $t \in N_0$. If

$$(3) \quad F^2(t) \leq c + A[t, p_1, \dots, p_{n-1}, pF],$$

for all $t \in N_0$, where $c \geq 0$ is a constant, then

$$(4) \quad F(t) \leq \sqrt{c} + \frac{1}{2}A[t, p_1, \dots, p_{n-1}, p],$$

for all $t \in N_0$.

Another useful discrete inequality is embodied in the following theorem.

THEOREM 2. Let $u(t) \geq 0, v(t) \geq 0, p_i(t) > 0$ for $i = 1, \dots, n-1$ and $h_j(t) \geq 0$ for $j = 1, 2, 3, 4$ be real-valued functions defined for $t \in N_0$. If c_1, c_2 and μ are non-negative constants such that

$$(5) \quad u^2(t) \leq c_1 + A[t, p_1, \dots, p_{n-1}, h_1 u] \\ + A[t, p_1, \dots, p_{n-1}, h_2 v],$$

$$(6) \quad v^2(t) \leq c_2 + A[t, p_1, \dots, p_{n-1}, h_3 u] \\ + A[t, p_1, \dots, p_{n-1}, h_4 v],$$

for all $t \in N_0$, where $\bar{u}(t) = \exp(-2\mu t)u(t)$ and $\bar{v}(t) = \exp(2\mu t)v(t)$ for $t \in N_0$, then

$$(7) \quad u(t) \leq \exp(\mu t) \left[\sqrt{2(c_1 + c_2)} + A[t, p_1, \dots, p_{n-1}, h] \right],$$

$$(8) \quad v(t) \leq \left[\sqrt{2(c_1 + c_2)} + A[t, p_1, \dots, p_{n-1}, h] \right],$$

for all $t \in N_0$, where

$$(9) \quad h(t) = \max\{[h_1(t) + h_3(t)], [h_2(t) + h_4(t)]\},$$

for $t \in N_0$.

We next establish the following discrete inequalities which can be used in some applications.

THEOREM 3. Let $F(x, y) \geq 0$, $g(x, y) \geq 0$ be real-valued functions defined for $x, y \in N_0$. If

$$(10) \quad F^2(x, y) \leq c + B[x, y, gF],$$

for all $x, y \in N_0$, where $c \geq 0$ is a constant, then

$$(11) \quad F(x, y) \leq \sqrt{c} + \frac{1}{2}B[x, y, g],$$

for all $x, y \in N_0$.

THEOREM 4. Let $u(x, y) \geq 0$, $v(x, y) \geq 0$ and $h_i(x, y) \geq 0$ for $i = 1, 2, 3, 4$ be real-valued functions defined for $x, y \in N_0$. If c_1, c_2 and μ are non-negative constants such that

$$(12) \quad u^2(x, y) \leq c_1 + B[x, y, h_1 u] + B[x, y, h_2 \bar{v}],$$

$$(13) \quad v^2(x, y) \leq c_2 + B[x, y, h_3 \bar{u}] + B[x, y, h_4 v],$$

for all $x, y \in N_0$, where $\bar{u}(x, y) = \exp(-2\mu(x+y))u(x, y)$ and $\bar{v}(x, y) = \exp(2\mu(x+y))v(x, y)$ for $x, y \in N_0$, then

$$(14) \quad u(x, y) \leq \exp(\mu(x+y)) \left[\sqrt{2(c_1 + c_2)} + B[x, y, h] \right],$$

$$(15) \quad v(x, y) \leq \left[\sqrt{2(c_1 + c_2)} + B[x, y, h] \right],$$

for all $x, y \in N_0$, where

$$(16) \quad h(x, y) = \max\{[h_1(x, y) + h_3(x, y)], [h_2(x, y) + h_4(x, y)]\},$$

for $x, y \in N_0$.

In the following theorems we establish the discrete inequalities similar to that given in Theorems 1 and 2, involving functions of several independent variables.

THEOREM 5. Let $F(x) \geq 0$, $b(x) \geq 0$, $b_i(x) > 0$ for $i = 1, \dots, n-1$ be real-valued functions defined for $x \in N_0^n$. If

$$(17) \quad F^2(x) \leq c + M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, bF],$$

for all $x \in N_0^n$, where $c \geq 0$ is a constant, then

$$(18) \quad F(x) \leq \sqrt{c} + \frac{1}{2}M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, b],$$

for all $x \in N_0^n$.

THEOREM 6. Let $u(x) \geq 0$, $v(x) \geq 0$, $b_i(x) > 0$ for $i = 1, \dots, n-1$ and $h_j(x) \geq 0$ for $j = 1, 2, 3, 4$ be real-valued functions defined for $x \in N_0^n$. If c_1, c_2 and μ be non-negative constants such that

$$(19) \quad u^2(x) \leq c_1 + M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, h_1u] \\ + M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, h_2\bar{v}],$$

$$(20) \quad v^2(x) \leq c_2 + M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, h_3\bar{u}] \\ + M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, h_4v],$$

for all $x \in N_0^n$, where $\bar{u}(x) = \exp(-2\mu \sum_{i=1}^n x_i)u(x)$ and $\bar{v}(x) = \exp(2\mu \sum_{i=1}^n x_i)v(x)$ for $x \in N_0^n$, then

$$(21) \quad u(x) \leq \exp(\mu \sum_{i=1}^n x_i) [\sqrt{2(c_1 + c_2)} + M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, h]]$$

$$(22) \quad v(x) \leq [\sqrt{2(c_1 + c_2)} + M[x_1, \dots, x_n, b_1, \dots, b_{n-1}, h]]$$

for all $x \in N_0^n$, where

$$(23) \quad h(x) = \max\{[h_1(x) + h_3(x)], [h_2(x) + h_4(x)]\},$$

for $x \in N_0^n$.

3. Proofs of Theorems 1 and 2

In order to establish the inequality (4) in Theorem 1, we first assume that $c > 0$ and define a function $z(t)$ by

$$(24) \quad z(t) = c + A[t, p_1, \dots, p_{n-1}, pF].$$

From (24) we observe that

$$(25) \quad L_n z(t) = p(t)F(t).$$

Using the fact that $F(t) \leq \sqrt{z(t)}$ and $\sqrt{z(t)} \leq \sqrt{z(t+1)}$ in (25) we observe that

$$(26) \quad \Delta \left[\frac{L_{n-1} z(t)}{\sqrt{z(t)}} \right] \leq p(t).$$

Now set $t = s_n$ in (26) and sum over $s_n = 0, 1, 2, \dots, t-1$ to obtain the estimate

$$(27) \quad \frac{L_{n-1} z(t)}{\sqrt{z(t)}} \leq \sum_{s_n=0}^{t-1} p(s_n).$$

Here we have used the fact that $L_{n-1} z(0) = 0$. Again as above, from (27) we observe that

$$(28) \quad \left[\frac{L_{n-2} z(t)}{\sqrt{z(t)}} \right] \leq p_{n-1}(t) \sum_{s_n=0}^{t-1} p(s_n),$$

from which we obtain the estimate

$$(29) \quad \frac{L_{n-2} z(t)}{\sqrt{z(t)}} \leq \sum_{s_{n-1}=0}^{t-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p(s_n).$$

Continuing in this way we obtain

$$(30) \quad \frac{\Delta z(t)}{\sqrt{z(t)}} \leq p_1(t) \sum_{s_2=0}^{t-1} p_2(s_2) \dots \sum_{s_{n-1}=0}^{s_{n-2}-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p(s_n).$$

Now we observe that

$$(31) \quad \begin{aligned} \Delta \sqrt{z(t)} &= \sqrt{z(t+1)} - \sqrt{z(t)} \\ &= \frac{\Delta z(t)}{\sqrt{z(t+1)} + \sqrt{z(t)}} \end{aligned}$$

$$\leq \frac{\Delta z(t)}{2\sqrt{z(t)}}.$$

Here in the last step we have used the fact that $\sqrt{z(t)} \leq \sqrt{z(t+1)}$. Using (30) in (31) we get

$$(32) \quad \Delta\sqrt{z(t)} \leq \frac{1}{2} p_1(t) \sum_{s_2=0}^{t-1} p_2(s_2) \dots \sum_{s_{n-1}=0}^{s_{n-2}-1} p_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} p(s_n).$$

Now set $t = s_1$ in (32) and sum over $s_1 = 0, 1, 2, \dots, t-1$ to obtain the estimate

$$(33) \quad \sqrt{z(t)} \leq \sqrt{c} + \frac{1}{2} A[t, p_1, \dots, p_{n-1}, p].$$

By using the fact that $F(t) \leq \sqrt{z(t)}$ in (33) we get the required inequality in (4).

Now suppose that $c = 0$. Then from (3) we see that the inequality

$$F^2(t) \leq \epsilon + A[t, p_1, \dots, p_{n-1}, p],$$

holds for every arbitrary small positive number ϵ and $t \in N_0$, which by the above argument yields the estimate

$$(34) \quad F(t) \leq \sqrt{\epsilon} + \frac{1}{2} A[t, p_1, \dots, p_{n-1}, p].$$

Since $F(t) \geq 0$ and $\epsilon > 0$ is arbitrary number independent on $t \in N_0$, then as $\epsilon \rightarrow 0$, it follows from (34) that

$$F(t) \leq \frac{1}{2} A[t, p_1, \dots, p_{n-1}, p].$$

This completes the proof of Theorem 1.

In order to prove Theorem 2, we first multiply (5) by $\exp(-2\mu t)$ and observe that

$$(35) \quad \{\exp(-\mu t)u(t)\}^2 \leq c_1 + A[t, p_1, \dots, p_{n-1}, h_1 \bar{u}] + A[t, p_1, \dots, p_{n-1}, h_2 v].$$

Define

$$(36) \quad F(t) = \exp(-\mu t)u(t) + v(t).$$

By squaring both sides of (36) and using the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$, (a, b reals), (35), (6) we observe that

$$(37) \quad \begin{aligned} F^2(t) &\leq 2[\{\exp(-\mu t)u(t)\}^2 + v^2(t)] \\ &\leq 2(c_1 + c_2) + 2A[t, p_1, \dots, p_{n-1}, [h_1 + h_3] \bar{u}] \\ &\quad + 2A[t, p_1, \dots, p_{n-1}, [h_2 + h_4]v]. \end{aligned}$$

Now by using the fact that $\exp(-2\mu t) \leq \exp(-\mu t)$ and (9) in (37) we observe that

$$(38) \quad F^2(t) \leq 2(c_1 + c_2) + 2A[t, p_1, \dots, p_{n-1}, hF].$$

The bounds in (7), (8) follow from an application of Theorem 1 to (38) and splitting. The proof of Theorem 2 is complete.

4. Proofs of Theorems 3-6

In order to establish the inequality (11) in Theorem 3, we first assume that $c > 0$ and define a function $z(x, y)$ by

$$(39) \quad z(x, y) = c + B[x, y, gF].$$

From (39) it is easy to observe that

$$(40) \quad \Delta_2^m[\Delta_1^n z(x, y)] = g(x, y)F(x, y).$$

Using the fact that $F(x, y) \leq \sqrt{z(x, y)}$ in (40), we have

$$(41) \quad \Delta_2^m[\Delta_1^n z(x, y)] \leq g(x, y)\sqrt{z(x, y)}.$$

From the definition of $z(x, y)$ we observe that $\sqrt{z(x, y)} \leq \sqrt{z(x, y+1)}$, for $x, y \in N_0$. Using this and the fact that $\Delta_2^{m-1}[\Delta_1^n z(x, y)] \geq 0$ in (41) we observe that

$$(42) \quad \frac{\Delta_2^{m-1}[\Delta_1^n z(x, y+1)]}{\sqrt{z(x, y+1)}} - \frac{\Delta_2^{m-1}[\Delta_1^n z(x, y)]}{\sqrt{z(x, y)}} \leq g(x, y).$$

Now keeping x fixed in (42), set $y = t_0$ and sum over $t_0 = 0, 1, 2, \dots, y-1$ and use the fact that $\Delta_2^{m-1}[\Delta_1^n z(x, 0)] = 0$, to obtain the estimate

$$(43) \quad \frac{\Delta_2^{m-1}[\Delta_1^n z(x, y)]}{\sqrt{z(x, y)}} \leq \sum_{t_0=0}^{y-1} g(x, t_0).$$

From (43) and using the facts that $\sqrt{z(x, y)} \leq \sqrt{z(x, y+1)}$ and $\Delta_2^{m-2}[\Delta_1^n z(x, y)] \geq 0$, we observe that

$$(44) \quad \frac{\Delta_2^{m-2}[\Delta_1^n z(x, y+1)]}{\sqrt{z(x, y+1)}} - \frac{\Delta_2^{m-2}[\Delta_1^n z(x, y)]}{\sqrt{z(x, y)}} \leq \sum_{t_0=0}^{y-1} g(x, t_0).$$

Keeping x fixed in (44), set $y = t_1$ and sum over $t_1 = 0, 1, 2, \dots, y-1$ and use the fact that $\Delta_2^{m-2}[\Delta_1^n z(x, 0)] = 0$, to obtain the estimate

$$(45) \quad \frac{\Delta_2^{m-2}[\Delta_1^n z(x, y)]}{\sqrt{z(x, y)}} \leq \sum_{t_1=0}^{y-1} \sum_{t_0=0}^{t_1-1} g(x, t_0).$$

Continuing in this way we obtain

$$(46) \quad \frac{\Delta_1^n z(x, y)}{\sqrt{z(x, y)}} \leq \sum_{t_{m-1}=0}^{y-1} \sum_{t_{m-2}=0}^{t_{m-1}-1} \cdots \sum_{t_0=0}^{t_1-1} g(x, t_0).$$

From (46) and using the facts that $\sqrt{z(x, y)} \leq \sqrt{z(x+1, y)}$ and $\Delta_1^{n-1} z(x, y) \geq 0$, we observe that

$$(47) \quad \begin{aligned} & \frac{\Delta_1^{n-1} z(x+1, y)}{\sqrt{z(x+1, y)}} - \frac{\Delta_1^{n-1} z(x, y)}{\sqrt{z(x, y)}} \\ & \leq \sum_{t_{m-1}=0}^{y-1} \sum_{t_{m-2}=0}^{t_{m-1}-1} \cdots \sum_{t_0=0}^{t_1-1} g(x, t_0). \end{aligned}$$

Now keeping y fixed in (47), set $x = s_0$ and sum over $s_0 = 0, 1, 2, \dots, x-1$ and use the fact that $\Delta_1^{n-1} z(0, y) = 0$, to obtain the estimate

$$(48) \quad \frac{\Delta_1^{n-1} z(x, y)}{\sqrt{z(x, y)}} \leq \sum_{s_0=0}^{x-1} \sum_{t_{m-1}=0}^{y-1} \sum_{t_{m-2}=0}^{t_{m-1}-1} \cdots \sum_{t_0=0}^{t_1-1} g(s_0, t_0).$$

Continuing in this way we obtain

$$(49) \quad \begin{aligned} \frac{\Delta_1 z(x, y)}{\sqrt{z(x, y)}} & \leq \sum_{s_{n-2}=0}^{x-1} \sum_{s_{n-3}=0}^{s_{n-2}-1} \cdots \sum_{s_0=0}^{s_1-1} \cdots \\ & \cdots \sum_{t_{m-1}=0}^{y-1} \sum_{t_{m-2}=0}^{t_{m-1}-1} \cdots \sum_{t_0=0}^{t_1-1} g(s_0, t_0). \end{aligned}$$

Now we observe that

$$(50) \quad \begin{aligned} \Delta_1 \sqrt{z(x, y)} & = \sqrt{z(x+1, y)} - \sqrt{z(x, y)} \\ & = \frac{z(x+1, y) - z(x, y)}{\sqrt{z(x+1, y)} + \sqrt{z(x, y)}} \\ & \leq \frac{\Delta_1 z(x, y)}{2\sqrt{z(x, y)}}. \end{aligned}$$

Here in the last step, we have used the fact that $\sqrt{z(x, y)} \leq \sqrt{z(x+1, y)}$. Using (49) in (50), we get

$$(51) \quad \Delta_1 \sqrt{z(x,y)} \leq \frac{1}{2} \sum_{s_{n-2}=0}^{x-1} \sum_{s_{n-3}=0}^{s_{n-2}-1} \dots \sum_{s_0=0}^{s_1-1} \dots \\ \dots \sum_{t_{m-1}=0}^{y-1} \sum_{t_{m-2}=0}^{t_{m-1}-1} \dots \sum_{t_0=0}^{t_1-1} g(s_0, t_0).$$

Now keeping y fixed in (51), set $x = s_{n-1}$ and sum over $s_{n-1} = 0, 1, 2, \dots, x-1$ and use the fact that $z(0, y) = c$, to obtain the estimate

$$(52) \quad \sqrt{z(x,y)} \leq \sqrt{c} + \frac{1}{2} B[x, y, g].$$

By using the fact that $F(x, y) \leq \sqrt{z(x, y)}$ in (51), we get the required inequality in (11). The proof of the case when $c=0$ can be completed by following the arguments as in the proof of Theorem 1 given above and hence the proof of Theorem 3 is complete.

The proof of Theorem 4 follows by the same arguments as in the proof of Theorem 2 and applying Theorem 3 with suitable modifications and hence we omit it here.

We note that the proofs of Theorems 5 and 6 are very close to that of the proofs of theorems given above and in view of the proofs of theorems given in [10]. We omit the details.

5. Some Applications

In this section we indicate some applications of our results to obtain upper bounds on the solutions of certain finite difference equations for which the earlier inequalities do not apply directly. For example, consider the following higher order nonlinear finite difference equation in two independent variables of the form

$$(53) \quad \Delta_2^m \Delta_1^n u^2(x, y) = f(x, y, u(x, y)),$$

with the given boundary conditions at $x=0$ and $y=0$

$$(54) \quad \Delta_2^j u^2(x, 0) = \alpha_j(x), \quad 0 \leq j \leq m-1,$$

$$(55) \quad \Delta_1^i u^2(0, y) = \beta_i(y), \quad 0 \leq i \leq n-1,$$

where $f: N_0 \times N_0 \times R \rightarrow R, \alpha_j, \beta_i: N_0 \rightarrow R$. Here R denotes the set of real numbers and

$m, n \geq 1$ are integers. The problem (53) - (55) can be represented by the equivalent difference equation

$$(56) \quad u^2(x, y) = b(x, y) + B[x, y, f(u)],$$

where $b(x, y)$ depends on the given boundary conditions (54), (55) and in (56) we have set $f(x, y, u(x, y)) = f(u)$. Suppose that the function f in (53) satisfies

$$(57) \quad |f(x, y, u)| \leq g(x, y) |u|,$$

where $g(x, y)$ is a real-valued non-negative function defined for $x, y \in N_0$ and assume further that the function $b(x, y)$ in (56) satisfies

$$(58) \quad |b(x, y)| \leq c,$$

where $c \geq 0$ is a constant. If $u(x, y)$ is a solution of the problem (53) - (55), then it satisfies the equation (56). Using (57), (58) in (56), we get

$$(59) \quad |u(x, y)|^2 \leq c + B[x, y, g |u|].$$

Now an application of Theorem 3 yields

$$(60) \quad |u(x, y)| \leq \sqrt{c} + \frac{1}{2}B[x, y, g].$$

The inequality (60) obtains the bound on the solution $u(x, y)$ of the problem (53) - (55) in terms of the known functions.

Further, it is to be noted that the inequality given in Theorem 4 can be used to obtain the bound on the solution of the following systems of finite difference equations of the form:

$$(61) \quad \Delta_2^m \Delta_1^n u^2(x, y) = f_1(x, y, u(x, y), v(x, y)),$$

$$(62) \quad \Delta_2^j u^2(x, 0) = a_j(x), \quad 0 \leq j \leq m-1,$$

$$(63) \quad \Delta_1^i u^2(0, y) = \beta_i(y), \quad 0 \leq i \leq n-1,$$

and

$$(64) \quad \Delta_2^m \Delta_1^n v^2(x, y) = f_2(x, y, u(x, y), v(x, y)),$$

$$(65) \quad \Delta_2^j v^2(x, 0) = \gamma_j(x), \quad 0 \leq j \leq m-1,$$

$$(66) \quad \Delta_1^i v^2(0, y) = \delta_i(y), \quad 0 \leq i \leq n-1,$$

under some suitable conditions on the functions involved in (61) - (63) and (64) - (66). For similar applications, see [8].

In concluding, we note that the results given in this paper have many possible applications to certain new classes of finite difference equations. For example, the inequalities presented here can be used to obtain error estimations of approximate solutions of certain ordinary and partial differential equations as well as for investigation of the convergence of the approximate solutions for such problems. However, the discussion of such applications of the developed inequalities is left to another place.

REFERENCES

1. E. F. Beckenbach and R. Bellman, "Inequalities", Springer-Verlag, Berlin, New York, 1961.
2. P. R. Beesack, "Gronwall Inequalities". Carleton University Mathematical Lecture Notes, No. 11, 1975.
3. D. E. Greene, An inequality for a class of integral systems, Proc. Amer. Math. Soc. 62 (1977), 101-104.
4. A. Haraux, "Nonlinear Evolution Equations: Global behavior of solutions", Lecture Notes in Mathematics No. 841, Springer-Verlag, Berlin, New York, 1981.
5. R. Ikehata and N. Okazawa, Yosida approximation and nonlinear hyperbolic equation, Nonlinear Analysis TMA 15 (1990), 479-495.
6. S. N. Olekhnik, Boundedness and unboundedness of solutions of some systems of ordinary differential equations, Vestnik Moskov. Univ. Mat. 27 (1972), 34-44.
7. L. Ou-Iang, The boundedness of solutions of linear differential equations $y'' + A(T)y = 0$, Shuxue Jinzhan 3 (1957), 409-415.

8. B. G. Pachpatte, A note on Greene's inequality, Tamkang Jour. Math. 15 (1984), 49-54.
 9. B. G. Pachpatte, On a class of new discrete inequalities, Tamkang Jour. Math. 20 (1989), 19-28.
 10. B. G. Pachpatte, On multidimensional discrete inequalities and their applications, Tamkang Jour. Math. 21 (1990), 111-122.
 11. B. G. Pachpatte, Discrete inequalities in two variables and their applications, Radovi Matematički 6 (1990), 235-247.
 12. M. Tsutsumi and I. Fukunda, On solutions of the derivative nonlinear Schrödinger equation. Existence and Uniqueness theorem, Funkcialaj Ekvacioj 23 (1980), 259-277.
-

