LIBERTAS MATHEMATICA, VOL. 13, 1993

OPTIMAL CONTROL OF AN ELLIPTIC GOVERNED PROBLEM

V. Barbu* and N. H. Pavel*

1. Introduction. We shall study the optimal control problem

$$\text{Minimize } L(y,u); \quad L(y,u) = \int_0^1 y(x,0)\,ds \tag{1.1}$$

subject to

$$y_{tt}(x,t) + (u(x)y_x(x,t))_x = f(x,t), \quad \text{in } (0,1)\times(0,T) = Q \tag{1.2}$$

$$y(x,0) = y(x,T); \quad y_t(x,0) = y_t(x,T) \tag{1.3}$$

$$y_x(0,t) = 0; \quad y(1,t) = 0. \tag{1.4}$$

The set U of admissible controls is the following:

$$\begin{aligned} U = \{&u \in \mathrm{Lip}\,(0,1); |u'(x)| \le \rho, \text{ a.e. in } (0,1) \\ &0 < a \le u(x) \le b,\ \forall x \in [0,1],\ u(0) = \alpha, u(1) = \beta\} \\ &\text{with } 0 < \alpha < \beta < b,\ \rho \ge \alpha + \beta - 2a > 0. \end{aligned} \tag{1.5}$$

Here $f \in L^2(Q)$ guarantees the existence and uniqueness of an $L^2(0,T;H)$ solution of (1.1)–(1.4), $H = L^2(0,1)$. However, we shall see that for the determination of the optimal control u^*, more regularity and sign conditions on f will be required.

The problem (1.2)–(1.4) can be viewed as a stationary heat conductor model for a strip conductor $(0,1)\times\mathbb{R}$ with periodic conditions in the variable t, with zero flux condition at $x = 0$ and zero temperature at $x = 1$. We shall determine the

Dedicated to Professor Roberto Conti with the occasion of his 70^{th} birthday.

* Supported by National Science Foundation under Grant DMS 91-11794.

optimal control u^* corresponding to the cost functional of the form (1.1). Other cost functionals can be treated in a similar way, but the determination of optimal control will be likely more complicated (or simply not possible). For example one can consider the following cost functionals $L_i(y,u)$, $i = 1,2$ in place of $L(y,u)$:

$$L_1(y,u) = \int_0^1 |y(x,0) - g_1(x)|^2 dx \qquad (1.1)'$$

or

$$L_2(y,u) = \int_0^T |y(0,t) - g_2(t)|^2 dt, \qquad (1.1)''$$

where $g_i \in L^2(0,k_i)$, $i = 1,2$; $k_1 = 1$, $k_2 = T$ are given functions. The minimization of L_1 and L_2 will be studied elsewhere. Note that the minimization of L (or L_i) is actually an identification problem, i.e.

How should we choose the coefficient u in the state equation (1.2) (under boundary condition (1.3)–(1.4)) to achieve some prescribed goal (e.g. to achieve the minimum of L (or L_i))?

In order to prove that for each $u \in U$, the problem (1.2)–(1.4) is well-posed, define:

$$D(A(u)) = \{H^2(0,1); y_x(0) = 0, y(1) = 0\} \qquad (1.6)$$

$$(A(u)y)(x) = (u(x)y_x(x))_x, y \in D(A(u)), u \in U, \qquad (1.7)$$

where $H^2(0,1)$ is the usual Sobolev space with $\Omega = (0,1)$.

Observe that $y(1) = 0$ yields

$$|y(x)| \leq \int_0^1 |y_x(s)|\, ds \leq |y_x|_{L^2(0,1)}, \quad y \in D(A(u)) \qquad (1.8)$$

so

$$|y|_{L^2(0,1)} \leq |y_x|_{L^2(0,1)}, \quad y \in D(A(u)). \qquad (1.9)$$

It is now easy to check that $A(u)$: $L^2(0,1) \to L^2(0,1)$ is strictly negative definite (strictly dissipative), namely

$$\langle A(u)y, y\rangle_{L^2(0,1)} = -\int_0^1 u(x)y_x^2(x)\, dx \leq -a|y_x|^2 \leq -a|y|^2, \forall y \in D(A(u)) \qquad (1.10)$$

with $a > 0$ as in (1.5) and $|y| = |y|_{L^2(0,1)}$. Moreover, $\langle A(u)y, z\rangle = \langle y, A(u)z\rangle_{L^2(0,1)}$ for all $y, z \in D(A(u))$ and therefore $A(u)$ is also selfadjoint.

Define as usual the realization $\tilde{A}$ of $A(u)$ in $L^2(0, T; H)$ by:

$$D(\tilde{A}) = \{\tilde{y} \in L^2(0, T; H) \colon \tilde{y}(t) \in D(A(u)), \text{ a.e. in } (0,T); t \to A(u)\tilde{y}(t) \in L^2(0, T; H)\}$$

and

$$(\tilde{A}\tilde{y})(t) = A(u)\tilde{y}(t), \text{ a.e. in } (0, T) \text{ for } \tilde{y} \in D(\tilde{A}) \tag{1.11}$$

with $\tilde{A} = \tilde{A}(u)$.

It is easy to check that $\tilde{A}$ is also strictly dissipative and selfadjoint in $L^2(0, T; H)$. Now, let us define:

$$D(B) = \{z \in H^2(0, T; H), z(0) = z(T), z_t(0) = z_t(T)\} \tag{1.12}$$

and

$$Bz = z_{tt}, \quad z \in D(B). \tag{1.13}$$

Then B is dissipative

$$\langle Bz, z\rangle_{L^2(0,T;H)} = -\int_0^T |z_t(s)|_H^2 ds \leq 0, \quad \forall z \in D(B), \tag{1.14}$$

and it is also known (and easy to check) that B is maximal dissipative [1].

On the other hand, the sum $\tilde{A} + B$ is a selfadjoint strictly negative definite operator in $L^2(0, T; H)$ so it is maximal dissipative onto and one-to-one. In other words, for every $f \in L^2(0, T; H)$, there is a unique $\tilde{y} \in D(\tilde{A}) \cap D(B)$ such that

$$B\tilde{y} + \tilde{A}(u)\tilde{y} = f \tag{1.15}$$

which means that $\tilde{y} = y(x, t)$ is the unique solution of (1.2)–(1.4) in $L^2(0, T; H)$. For more details we refer to [1]. Summarizing, we have proved:

Theorem 1.1. *For every $u \in U$ and $f \in L^2(Q)$, the problem* (1.2)–(1.4) *has a unique solution $y = y^u \in D(\tilde{A}) \cap C(B)$.*

2. Determination of optimal control. In order to define precisely the optimal control u^* for Problem (1.1), introduce the set M of admissible pairs (y, u).

$$M = \{(y, u), y \in D(\tilde{A}(u)) \cap D(B), u \in U, (y, u) \text{ related as in } (1.15)\}. \tag{2.1}$$

In other words $(y, u) \in M$ means: y is the $L^2(Q)$ solution of (1.2)–(1.4) corresponding to $u \in U$.

By definition, an optimal pair $(y^*, u^*) \in M$ satisfies

$$L(y^*, u^*) = \inf\{L(y, u), (y, u) \in M\} \tag{2.2}$$

with L as in (1.1). Here u^* is an optimal control and $y^* = y_{u^*}$ is the corresponding optimal arc.

Now let us rewrite (1.2)–(1.4) as

$$y_{tt} + A(u)y = f \quad \text{in } L^2(0, T; H) = L^2(Q) \tag{2.3}$$

$$y(0) = y(T), \quad y_t(0) = y_t(T) \tag{2.4}$$

and introduce the tangent cone to M at $(y, u) \in M$:

$$\begin{aligned} \operatorname{Tan} M(y, u) = \{&(v, w), v \in L^2(Q); w \in \operatorname{Tan} U(u); \\ &v_{tt} + A(u)v + A(w)y = 0 \quad \text{in } L^2(Q) \\ &v(0) = v(T); \quad v_t(0) = v_t(T)\}. \end{aligned} \tag{2.5}$$

The existence of an optimal pair (y^*, u^*) follows by standard arguments [3] and it is not our goal here. It is our purpose here to determine explicitly the optimal control u^* (under additional restrictions on f). Precisely the main result of this paper is given by:

Theorem 2.1. *Let $f \in C^1(Q)$ satisfy the following sign condition:*

$$f(x,t) > 0, \quad f_x(x,t) < 0 \quad \text{in } Q. \tag{2.6}$$

Then the optimal control u^* of Problem (1.1) is given by:

$$u^*(x) = \begin{cases} \alpha - \rho x, & \text{for } x \in [0, (\alpha - a)\rho^{-1}] \\ a, & \text{for } x \in [(\alpha - a)\rho^{-1}, (a + \rho - \beta)\rho^{-1}] \\ \beta + \rho(x - 1), & \text{for } x \in [(a + \rho - \beta)\rho^{-1}, 1] \end{cases} \tag{2.7}$$

with a, α, β and ρ as in (1.5).

The proof of this theorem is delicate and relies on the lemmas below.

Lemma 2.1. *Let $f \in C^1(Q)$ satisfy the sign condition (2.6). Then the solution y of (2.3)-(2.4) satisfies the sign condition*

$$y(x,t) < 0, \quad y_x(x,t) > 0 \quad \text{in } Q.$$

Proof. Since $f > 0$ in Q, (1.2) yields

$$y_{tt}(x,t) + (u(x)y_x(x,t))_x > 0 \quad \text{in } Q,$$

so according to the maximum principle for elliptic equations, we have

$$\max_Q y(x,t) = \max_{\partial Q} y(x,t). \tag{2.8}$$

Maximum of y cannot be assumed at a point $(0,t)$ of the boundary ∂Q of Q. Indeed, if we assume by contradiction that $\max_{\partial Q} y(x,t) = y(0,t_0)$, $0 < t_0 < T$, then the normal derivative $\frac{\partial y}{\partial v_x}(0,t_0) > 0$. But $\frac{\partial y}{\partial v_x}(0,t_0) = -y_x(0,t_0) = 0$ which is not the case (as $y_x(0,t_0) = 0$ by (1.3)). Similarly we can check that the situation

$$\max_{\partial Q} y(x,t) = y(x_0,0) = y(x_0,T), \quad 0 < x_0 < 1 \tag{2.9}$$

is also impossible. Indeed, (2.9) would imply

$$\frac{\partial y}{\partial v_t}(x_0,0) > 0, \quad \frac{\partial y}{\partial v_t}(x_0 T) > 0 \tag{2.10}$$

$$-y_t(x_0,0) > 0 \quad \text{and} \quad y_t(x_0,T) > 0 \tag{2.11}$$

which is in conflict with (1.3). Therefore

$$\max_{\partial Q} y(x,t) = \max_{0 \le t \le T} y(1,t) = 0$$

so $y(x,t) < 0$ in Q. To complete the proof, set:

$$z(x,t) = u(x)y_x(x,t) \quad \text{in } Q. \tag{2.12}$$

In view of (1.2)-(1.4) and $y(x,t) < 0$ in Q we derive:

$$z_{tt} + u(x)z_{xx} = u(x)f_x(x,t) < 0 \quad \text{in } Q \tag{2.13}$$

$$z(x,0) = z(x,T); \quad z_t(x,0) = z_t(x,T), \quad \text{in } (0,1) \tag{2.14}$$

$$z(0,t) = 0; \quad z(1,t) \ge 0 \quad \text{in } (0,T) \tag{2.15}$$

so, according to minimum principle,

$$\min_{Q} z(x,t) = \min_{\partial Q} z(x,t). \tag{2.16}$$

Arguing as above, z cannot assume its minimum (infimum) on the subboundary $\{(x,0),(x,(T)), 0 < x < 1\}$ of ∂Q (due to the periodic condition (2.14)). Therefore $\min_{\partial Q} z(x,t) = 0$ (by (2.15)), so $z(x,t) > 0$ in Q, i.e. $y_x(x,t) > 0$ in Q. This completes the proof. ☐

The most delicate result necessary in the proof of Theorem 2.1 is given by

Lemma 2.1. *A function $w \in L^1(0,1)$ belongs to the normal cone $N_U(u)$ to U at $u \in U$ if and only if:*

$$w = -\theta' + \eta \quad \text{in } \mathcal{D}'(0,1) \tag{2.17}$$

with $\theta \in L^1(0,1)$, $\eta \in L^1(0,1)$ satisfying

$$\theta(x) = 0 \text{ a.e. in } \{x \in (0,1); |u'(x)| < \rho\}$$

$$\theta(x) = \lambda(x)u'(x) \text{ a.e. in } \{x \in (0,1); |u'(x)| = \rho\},$$

where $\lambda \in L^1(0,1)$, $\lambda(x) \geq 0$ a.e. in $(0,1)$.

$$\eta(x) = 0, \text{ a.e. in } \{x \in (0,1); a < u(x) < b\}$$

$$\eta(x) \leq 0, \text{ a.e. in } \{x \in (0,1); u(x) = a\}$$

$$\eta(x) \geq 0, \text{ a.e. in } \{x \in (0,1); u(x) = b\}.$$

The proof of this important result can be found in [2], [4] or [5].

Proof of Theorem 2.1. Let $(y^*, u^*) \in M$ be an optimal pair for problem (1.1) and let p be the solution of the following dual problem of (1.1)–(1.4)

$$p_{tt} + (u^*(x)p_x(x,t))_x = 0 \quad \text{in } Q \tag{2.17}$$

$$p_x(0,t) = 0, \quad p(1,t) = 0 \quad \text{in } [0,T] \tag{2.18}$$

$$p(x,0) = p(x,T); \quad p_t(x,T) - p_t(x,0) = -1 \quad \text{in } (0,1). \tag{2.19}$$

The existence and uniqueness of the solution p can be proved as in the case of (1.2)–(1.4). The only adjustment is that in this case

$$D(B) = \{z \in H^2(0,T;H), z(0) = z(T), z_t(0) - z_t(T) = 1\} \tag{2.20}$$

so B is a nonlinear maximal dissipative operator in $L^2(0,T;H)$. It follows that $\tilde{A}+B$ is also onto and one-to-one from $D(\tilde{A})\cap D(B)$ into $L^2(0,T;H)$ so there is a unique solution $p\in D(\tilde{A})\cap D(B)$ of Problem (2.17)–(2.19). We now prove that

$$p(x,t)<0 \quad \text{and} \quad p_x(x,t)>0 \quad \text{in } Q. \tag{2.21}$$

Indeed, according to maximum principle

$$\max_Q p(x,t)=\max_{\partial Q} p(x,t). \tag{2.22}$$

As in the proof of Lemma 2.1, the maximum of $p(x,t)$ cannot be assumed on the subboundary $\{(0,t),0<t<T\}=S_0$ of ∂Q. In view of (2.19) we can also check that $\max_{\partial Q} p(x,t)$ cannot be assumed on $\{(x,0),(x,T),0<x<1\}=S_T$. Indeed, if $\max_{\partial Q} p(x,t)=p(x_0,0)$ then

$$\frac{\partial p}{\partial v}(x_0,0)>0,\quad \frac{\partial p}{\partial v}(x_0,T)>0$$

which means

$$-p_t(x_0,0)>0,\quad p_t(x_0,T)>0 \tag{2.23}$$

respectively. Or, (2.23) is in conflict with (2.19). Therefore the only possibility is

$$\max_Q p(x,t)=p(1,t)=0$$

so $p(x,t)<0$ in Q. The proof of the inequality $p_x(x,t)>0$ in Q follows the same steps as the proof of $y_x(x,t)>0$ in Q, with $z(x,t)=u^*(x)p_x(x,t)$ and therefore (2.21) holds.

We are now in a position to complete the proof.

Recall that if

$$G(F)=\inf_{y\in M} G(y)$$

with $G\colon M\to\mathbb{R}$ — Frechet differentiable, then

$$\partial G(F)(Y)\geq 0 \text{ for all } Y\in \operatorname{Tan} M(Y), \tag{2.24}$$

where $\partial G(F)$ stands for the Frechet derivative of G at $F\in M$.

In view of this simple principle, we can derive (in the case of $G(y) = L(y, u)$):

$$\int_0^1 v(x,0)\,dx \geq 0 \text{ for all } (v,w) \in \operatorname{Tan} M(y^*, u^*). \tag{2.25}$$

Multiplying (2.17) by v and integrating over $[0,T]$ and $[0,1]$ one obtains successively:

$$\langle p_t, v\rangle|_0^T - \int_0^T \langle p_t, v_t\rangle\,dt + \int_0^T \langle A(u^*)p(t), v(t)\rangle\,dt = 0 \tag{2.26}$$

with $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx = \langle f,g\rangle_H$, $H = L^2(0,1)$. Therefore

$$-\int_0^1 v(x,0)\,dx + \int_0^T \langle v_{tt} + A(u^*)v, p\rangle_H dt = 0 \tag{2.27}$$

with v as in (2.5), so

$$-\int_0^1 v(x,0)dx + \int_0^1 \int_0^T w(x) y_x^*(x,t) p_x(x,t)\,dx\,dt = 0 \tag{2.28}$$

which implies:

$$\int_0^1 w(x)\left(\int_0^T y_x^*(x,t)p_x(x,t)\,dt\right)dt = \int_0^1 v(x,0)\,dx \geq 0$$

i.e. the function φ defined by

$$\varphi(x) = -\int_0^T y_x^*(x,t)p_x(x,t)\,dx, \quad x \in (0,1) \tag{2.29}$$

belongs to the normal cone $N_U(u^*)$ to U at u^*. One the other hand by Lemma 2.1 and (2.21), $\varphi(x) < 0$ in $[0,1]$. Or, it was proved in [2, 4, 5] that if $N_U(u^*)$ contains a negative function, then u^* is convex. In short, this can be proven as below: Since $u^*(0) = \alpha > a$ the open set $M_a = \{x \in (0,1); u^*(x) > a\}$ is nonempty. Taking into account (2.17), it follows

$$-\theta' + \eta = \varphi < 0 \quad \text{with} \quad \eta \geq 0 \text{ in } M_a$$

so $-\theta' < 0$ in M_a. This implies that $|\frac{d}{dx}(u^*)| = \rho$ in M_a. Next one proves that u^* is convex so the only possibility for u^* is to have the form given by (2.7). This completes the proof. ☐

Note that there is a growing interest in the optimal control of elliptic equations. See e.g. the recent paper [6].

The optimization of L_1 and L_2 defined by $(1.1)'$ and $(1.1)''$ remains an open problem (to be studied elsewhere).

References

[1] S. Aizicovici and N. H. Pavel, *Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space*, J. Funct. Anal., 99 (1991), 387–408.

[2] V. Barbu, "Mathematical Methods for Optimization of Differential Systems," in preparation, D. Reidel Publ. Co.

[3] V. Barbu and N. H. Pavel, *Optimal control problems with two point boundary condition*, J. Optimiz. Th. Appl., 77 (1993), 51–78.

[4] V. Barbu and N. H. Pavel, *Optimal control of thermal conductivity of a rod under periodic conditions*, to appear.

[5] A. Friedman, *Nonlinear optimal control for parabolic equations*, SIAM J. Control Optimiz., 22 (1982), 805–816 (see also 27 (1987)).

[6] R. Tahraoui, *Contrôl optimal dans les équations elliptiques*, SIAM J. Control Optimiz., 30 (1992), 495–521.