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To Professor Roberto Conti,
with warmest affection and heartfelt gratitude

SOME MOUNTAIN CLIMBING TECHNIQUES IN THE THEORY OF SEMILINEAR OPERATOR EQUATIONS

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Related to the mountain pass lemma, recently have appeared the mountain impasse and mountain cliff theorems; this is the reason for the chosen title. It is the purpose of our research to discuss some alternatives regarding the number and the nature of solutions of semilinear operator equations by using the mountain pass strategy without the Palais - Smale condition. Our paper has its root in some recent contributions by M.Schechter [11] - [13].

Many variational problems can be formulated in such a way that one is seeking a solution of

(0) $$F'(u) = 0, \quad u \neq 0$$

for a C^1 functional F defined on a real Banach space X. When F is unbounded both from above and below, it is extremely difficult to find a local stationary point of F which will produce a solution of (0). One successful method is the mountain pass lemma (MPL) of Ambrosetti - Rabinowitz [1].

The starting result in its simple form is

Mountain Pass Lemma. Let $(X, \|\cdot\|)$ be a real Banach space and $F: X \longmapsto \mathbb{R}$ be a continuously Fréchet differentiable functional. Assume that

(A) $F(0) = 0$ and there are constants $\rho, \alpha > 0$ such that $F(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$;

(B) There is an $e \in X$ such that $\|e\| > \rho$ and $F(e) < \alpha$.

Let

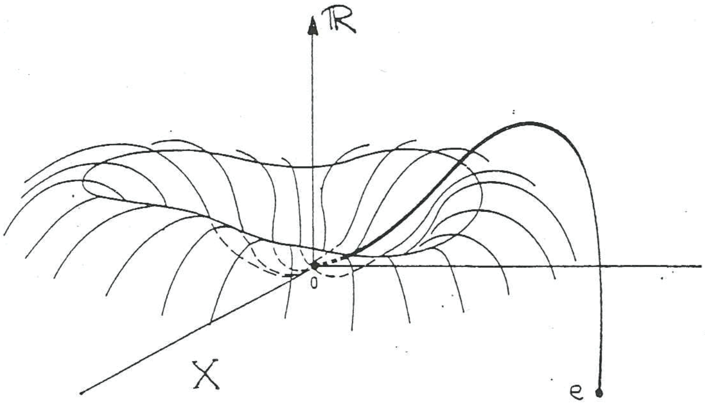
$$c = \inf_{p \in \Gamma_0} \max_{0 \le t \le 1} F(p((t))$$

where

$$\Gamma = \{\, p \in C^0([0,1],X) \mid p(0) = 0, \ p(1) = e \,\}.$$

If F satisfies the Palais - Smale condition (PS_c), then c is a critical point of F.

In an intuitive view, the theorem says if a pair of points in the graph of F are separated by a mountain range, there must be a mountain pass containing a critical point between them. Obviously, $c \geq \alpha$.



This existence result for free critical points was established by A.Ambrosetti and P.H.Rabinowitz [2], using the Morse deformation method. A natural proof was given then by means of Ekeland's variational principle [5].

§ 1. THE EKELAND VARIATIONAL PRINCIPLE

The variational principle states that if a differentiable functional F on a Banach space X attains its minimum at some point u, then $F'(u) = 0$; it has been proved a valuable tool for studying PDEs.The Ekeland principle shows that if F has a finite lower bound (although it not need attain it), then, for every $\varepsilon > 0$ there exists some point $u_\varepsilon \in X$ where $\|F'(u_\varepsilon)\|_{X^*} \leq \varepsilon$, i.e., its derivatives can be made arbitrarily small.

The ε-variational principle. Let (X,d) be a complete metric space and let $F:X \longmapsto R \cup \{+\infty\}$ be a l.s.c. functional, $\not\equiv +\infty$, bounded from below. Then, for every $x_0 \in X$ and $\varepsilon > 0$ satisfying

$$F(x_0) < \inf_{x \in X} F + \varepsilon$$

there exist $x_\varepsilon \in X$ such that

$$F(x_\varepsilon) + \varepsilon\, d(x_\varepsilon,x) \leq F(x_0) \tag{1.1}$$

$$F(x) \geq F(x_\varepsilon) - \varepsilon d(x,x_\varepsilon), \qquad \forall\, x \neq x_\varepsilon . \tag{1.2}$$

Formula (1.1) usually splits in

$$F(x_\varepsilon) < F(x_0) + \varepsilon, \qquad d(x_\varepsilon,x_0) < 1.$$

and so x_ε improves x_0 from the point of view of minimization and it is located in a neighborhood of x_0. Moreover, (1.2) says that the downward slope of F in x_0 is smaller than ε, because

$$\sup_{x \neq x_\varepsilon} \max \left(\frac{F(x_\varepsilon) - F(x)}{d(x,x_\varepsilon)} , 0 \right) \leq \varepsilon .$$

and so the downward slope vanishes if x_ε is a minimum of F. In particular, when X is a Banach space and F is differentiable, (1.2) implies

$$\| F(x_\varepsilon) \|_{X^*} \leq \varepsilon \ ;$$

thus the derivatives is almost flat as ε is small.

Replacing d with $d' = \frac{1}{\lambda}\, d$ with any $\lambda > 0$, (X,d') remains still a complete metric space. The above principle to (X,d') gives a point $y_\lambda \in X$ such that

$$d(y_\lambda,x_0) \leq \lambda$$

$$F(x) > F(y_\lambda) - \frac{\varepsilon}{\lambda}\, d(x,y_\lambda), \qquad \forall\, x \neq y_\lambda .$$

We may take $\lambda = \sqrt{\varepsilon}$ and obtain

Ekeland Principle. Let (X,d) be a complete metric space and let $F:X \longmapsto \mathbb{R} \cup \{+\infty\}$ be a l.s.c. functional, $\not\equiv +\infty$, with $\inf F > -\infty$. Then, for every $x_0 \in X$ and $\varepsilon > 0$, satisfying

$$F(x_0) \leq \inf_{x \in X} F + \varepsilon$$

there exist $x_\varepsilon \in X$ such that

$$F(x_\varepsilon) \leq F(x_o)$$

$$d(x_\varepsilon, x_o) \leq \sqrt{\varepsilon}$$

$$F(x) > F(x_\varepsilon) - \sqrt{\varepsilon}\, d(x, x_\varepsilon), \quad \forall\, x \neq x_\varepsilon .$$

Later on, X is a Banach space and F is a continuously Fréchet dfferentiable function, i.e. a C^1-function. The sequence $\{u_n\}$ is minimizing for F if $F(u_n) \longrightarrow \operatorname{Inf} F$.

Further, we assume the following weak "compactness condition":

(PS_c) Any sequence $\{u_n\}$ in X such that $F(u_n) \longrightarrow c$ and $\|F'(u_n)\|_{X^*} \longrightarrow 0$ has a convergent subsequence.

If this holds for every $c \in \mathbb{R}$, we say that F satisfies (PS). In general, it turned out that the (PS) condition suffices in current minimization procedures.

COROLLARY 1.1. If F is bounded from below and satisfies (PS), then every minimizing sequence has a convergent subsequence.

A function F is called <u>coercive</u> if $F(u) \longrightarrow \infty$ as $\|u\| \longrightarrow \infty$.

COROLLARY 1.2. If F is bounded from below and satisfies (PS), then F is is coercive.

A slightly stronger form of Ekeland's principle says [8]:

Let F be a C^1-function on a Banach space X. Assume that for every continuous path γ joining 0 and e in X we have

$$F(0),\ F(e) < \max_{\gamma} F.$$

Define

$$c = \inf_{\gamma} \max_{u \in \gamma} F(u),$$

over all possible paths γ. Then, for every $\varepsilon > 0$, there exists a path $\bar{\gamma}$ and a point $\bar{u} \in \bar{\gamma}$, where F achieves the maximum on $\bar{\gamma}$, such that $F(u) < c + \varepsilon$ and $\|F'(u)\| < \varepsilon$.

§ 2. A generalized mountain pass lemma

Let X be a Banach space and $F:X \longrightarrow \mathbb{R}$ be a C^1 function satisfying the condition:

(C) There is an open neighborhood U of 0 and some point $e \notin U$ such that

$$F(0),\ F(e) < \alpha \le F(u) \quad \forall\ u \in \partial U.$$

Let K be a compact metric space and let $K_o \subsetneqq K$ be a nonempty closed subset. We introduce the family

$$\Gamma = \{\ p \in C(K,X) \mid p = p_o \ \text{on}\ K_o\ \},$$

where $C(K,X)$ denotes the set of all continuous maps from K into X and p_o is a fixed element of $C(K,X)$. Define

$$c = \inf_{p \in \Gamma} \max_{\xi \in K} F(p(\xi)) \tag{2.1}$$

and assume that

$$\max_{\xi \in K} F(p(\xi)) > \max_{\xi \in K_o} F(p(\xi)) \qquad \forall\ p \in \Gamma. \tag{2.2}$$

An equivalent hypothesis of (2.2) is that for every $p \in \Gamma$ there is some point $\xi \in K \setminus K^*$ such that $F(p(\xi)) \ge c$.

The basic result in minimax procedures is given by

LEMMA 2.1. In the above setting, there exists a sequence $\{u_n\}$ in X, such that

$$F(u_n) \longrightarrow c \quad \text{and} \quad \|F'(u_n)\| \longrightarrow 0. \tag{2.3}$$

SKETCH OF THE PROOF [4]. For $x \in K$, set $d(\xi) = \min\{\operatorname{dist}(\xi, K_o), 1\}$ and, for any fixed $\varepsilon > o$ and $p \in \Gamma$, consider the perturbing function

$$G(p,\xi) = F(p(\xi)) + \varepsilon\, d(\xi)$$

Set

$$\psi_\varepsilon(p) = \max_{\xi \in K} G(p(\xi),\xi) \quad \text{and} \quad c_\varepsilon = \inf_{p \in \Gamma} \psi_\varepsilon(p).$$

Consider Γ as a metric space endowed with the uniform topology. Since the function $G:\Gamma \times K \longrightarrow \mathbb{R}$ is continuous, then ψ_ε is lower semicontinuous. Obviously, $\alpha \le c \le c_\varepsilon \le c + \varepsilon$ and so ψ_ε is bounded from below. By applying the ε-principle, for every $\varepsilon > o$, we obain the existence of $p_\varepsilon \in \Gamma$ such that

$$
(2.4)\qquad \begin{aligned} &\psi_\varepsilon(p_\varepsilon) \leq c_\varepsilon + \varepsilon \\ &\psi_\varepsilon(p) \geq \psi_\varepsilon(p_\varepsilon) - \varepsilon\|p - p_\varepsilon\|_{C_\circ} \quad \forall\, p \in \Gamma. \end{aligned}
$$

Consider now the set

$$
B_\varepsilon(p) = \{\ \xi \in K \mid G(p(\xi),\xi) = \psi_\varepsilon(p)\ \}.
$$

As, $\psi_\varepsilon(p) > \max\limits_{\xi \in K_0} F(p(\xi))$, by hypothesis (2.3), it follows that $B_\varepsilon(p) \subset K \setminus K_0$.

On the other hand, we can deduce from (2.4) that there exists at least one $\xi_\varepsilon \in K$ such that $\|F'(p_\varepsilon(\xi_\varepsilon)\| \leq \varepsilon$. Finally, it suffices to take $u_\varepsilon = p_\varepsilon(\xi_\varepsilon)$ in (2.4) and $\varepsilon = \frac{1}{n}$. ■

THEOREM 2.1. If in addition F satisfies (PS_c), then c, determined by (2.1), is a critical value.

The original MPL is clearly a particular case when $K = [0,1]$, $K = \{0,1\}$ and $p_0(t) = te$.

§ 3. A Pattern

The (PS) condition allows us to deal with unbounded regions in a uniform way. A drawback is that the (PS) condition is rather restrictive in applications. For example, we consider the Dirichlet problem

$$
(3.1)\qquad -\Delta u = g(x,u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0
$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $g(x,s)$ is a Carathéodory function in $\bar{\Omega} \times \mathbb{R}$. With the problem (3.1) we associate the functional

$$
F(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega G(x,u)\,dx
$$

where

$$
G(x,t) = \int_0^t g(x,s)\,ds
$$

and $\|\cdot\|_2$ is the norm in $L^2(\Omega)$. Let $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to

$$
\|u\| = \left[\int_\Omega |\nabla u|^2 dx\right]^{\frac{1}{2}}
$$

If $p(x,s)$ satisfies the growth condition

(3.2) $$|g(x,s)| \leq c|s|^{p-1} + h(x), \quad \forall\ (x,s) \in \Omega \times \mathbb{R},$$

with a constant $c > 0$, $h \in L^{p'}(\Omega)$, and $p < \frac{2n}{n-2}$ if $n \geq 3$, $\frac{1}{p} + \frac{1}{p'} = 1$.

In this case, the functional F is continously Fréchet differentiable on $H^1_0(\Omega)$ or $F \in C^1$ and

$$\langle F'(u),v\rangle = \int_\Omega \nabla u.\nabla v\ dx - \int_\Omega g(x,u)v\ dx \quad \forall\ v \in H^1_0(\Omega)$$

where $\langle\cdot,\cdot\rangle$ stands for product in $H^1_0(\Omega)$. Thus u is a critical point of F if and only if u is a weak solution of (3.1).

We further assume that

(3.3) $$g(x,s) = o(|s|) \text{ uniformly in } x \text{ as } s \to 0.$$

Moreover, it has been proved ([10]) that a sufficient hypothesis in order that F satisfy the (PS) condition is: there is a $\mu > 2$ such that

(3.3) $$0 < \mu\, G(x,s) \leq sg(x,s), \quad x \in \Omega, \quad |s| \text{ large}.$$

Integrating the last inequality, we reach an estimate of the form

(3.4) $$G(x,s) \geq a|s|^\mu - b$$

where a,b are positive constants, and (3.3) can be regarded as a coerciveness condition. By using condition (3.4), we can readily show that $F(tu) \longrightarrow -\infty$ as $t \to \infty$ for any $u \neq 0$. Hence there exists $0 \neq e \in X$ such that

(3.5) $$F(e) < 0$$

and so the condition (B) in the MPL is verified on some ball as well. Therefore, under hypotheses (3.2) - (3.3), the problem (3.1) has nontrivial weak solutions. In particular, (3.2) and (3.3) with $2 < \mu < p$ imply

$$a_0|u|^\mu \leq ug(x,u) \leq a_1|u|^p$$

and thereby the asymptotic behavior of $g(x,u)$ is confined to a rather narrow range.

M.Schechter [11] - [12] studied the cases when the condition (3.1),(3.3), (3.5) hold but (3.4) does not. The solvability of the Dirichlet problem for semilinear elliptic is replaced by certain alternatives between the number of solutions of the problem (3.1) and the number of solutions of the operator equation

$$(3.6) \qquad -\Delta u = \lambda\, g(x,u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \|\nabla u\| = R, \quad o < \lambda < 1.$$

for any number $R > \|e\|$.

Subsequently, we will remove the (PS) condition. More precisely, we will use weaker and local compactness conditions for the semilinear equations on bounded domains.

A similar procedure is used in [14] to prove the existence of nontrivial solutions for Hammerstein equations of the form $(I - KG)u = 0$. Sufficient spectral conditions of the operator KG that the corresponding functional satisfy the condition (PS) are estabished.

§ 4. The Morse deformation method

For an easier understanding of the subsequent treatment, we recall some basic results of the deformation procedure.

Let X be a real Banach space with the norm $\|\cdot\|$, $U \subset X$ and $F \in C^1(U,\mathbb{R})$. We consider the level set $F_a = \{x \in X \mid F(x) \leq a\}$ and let K_a be the set of critical points u of F where $F(u) = a$. Denote $Y = \{x \in X \mid F'(x) \neq 0\}$. A locally Lipschitz vector field $v: Y \longmapsto X$ is called a *pseudo-gradient* if for every $u \in U$ we have

$$\|v(x)\| \leq 2\|F'(x)\| \qquad \text{and} \qquad \langle F'(x),v(x)\rangle \geq \|F'(x)\|^2 .$$

The concept of pseudo-gradint has a key role in the establishing of the

GENERAL DEFORMATION THEOREM [4]. Let $c \in R$. For any given $\delta < \frac{1}{8}$ there exists a continuous deformation $\eta:[0,1] \times X \rightarrowtail X$ such that

1°. $\eta(0,x) = x$ for all $x \in X$;

2°. $\eta(t,\cdot)$ is a homeomorphism of X onto X for each $t \in [0,1]$;

3°. $\eta(t,x) = x$ for all $t \in [0,1]$ if $|F(x) - c| \geq 2\delta$ or if $\|F'(x)\| \leq \sqrt{\delta}$;

4°. $0 \leq F(x) - F(\eta(t,x)) \leq 4\delta$ for all $t \in [0,1]$ and $x \in X$;

5°. $\|\eta(t,x) - x\| \leq 16\sqrt{\delta}$ for all $t \in [0,1]$ and $x \in X$;

6°. Let $\tau \in [0,1]$ and assume that, for all $t \in [0,1]$, $\eta(t,x)$ belongs to the set $\{v \in X \mid |F(v) - c| \leq \delta \text{ and } \|F'(v)\| \geq 2\sqrt{\delta}\}$ then $F(\eta(\tau,x)) \leq F(x) - \frac{\tau}{4}$.

COROLLARY 2.1. If, in addition, F satisfies the condition $(PS)_c$, then for given $\varepsilon > 0$ there exists $\delta < \varepsilon$ and a deformation η as above such that

7°. if $x \in F_{c+\delta}$ and $F(\eta(1,x)) > c-\delta$ then $\| F'(\eta(t,x))\| < \varepsilon \quad \forall\, t \in [0,1]$.

COROLLARY 2.2. Suppose that F satisfies the condition $(PS)_c$. For given $\varepsilon > 0$ and a neighborhood $\mathcal{O}$ of K_c, there exist $\delta < \varepsilon$ and a deformation η as above such that

8°. $\eta(1, F_{c+\delta} \setminus \mathcal{O}) \subset F_{c-\delta}$.

Once this deformation theorem proved, the original MPL easily follows [5].

In the sequel, we are concerned with the bounded mountain pass lemma. Let H be a real Hilbert space and we denote by B_R the closed ball of radius R centred in the origin. To prevent that the paths to leave this ball, we seal the border $S_R = \partial B_R$ and consider functions $F \in C^1(B_R, \mathbb{R})$ satisfying the following hypothesis:

a). There is an element $0 \neq e \in B_R$ and $0 < \rho < \|e\|$ such that

$$F(0),\ F(e) < \alpha = \inf_{\|u\|=\rho} F(u).$$

Let Γ denote the set of continuous maps $\varphi : [0,1] \longmapsto B_R$ such that $\varphi(0) = 0$ and $\varphi(1) = e$. We define

$$c = c_R = \inf_{\varphi \in \Gamma} \max_{0 \leq s \leq 1} F(\varphi(s))$$

We consider the following MPL assuring that the paths will not leave a bounded domain. The boundary conditions impose in the case of a ball B_R require that there are constants $\varepsilon > 0$ and $\theta < 1$ such that

$$(F'(u),u) \geq -\theta \|F'(u)\| \|u\| \tag{4.1}$$

whenever $u \in S_R$ satisfies

$$|F(u) - c| \leq 3\varepsilon . \tag{4.2}$$

The proof of a deformation theorem corresponding to a function satisfying condition (4,1)-(4.2) relies onthe following generalization of the concept of pseudogradient.

LEMMA 4.1. Let $0 \neq Z \in C^1(E,H)$ where $E = B_R \setminus \{0\}$. Assume that there is a closed subset Q of E and $\theta < 1$ such that

$$(Z(u),u) + \theta \, \|Z(u)\| \, \|u\| \geq 0 , \quad \forall \quad u \in Q.$$

Then for each $\alpha < 1 - \theta$ there exists a locally Lipschitz map $Y:E \longrightarrow H$ with properties

$$(Z(u),Y(u)) \geq \alpha \, \|Z(u)\| , \quad u \in E$$

$$(Y(u),u) > 0, \quad u \in Q$$

and

$$\|Y(u)\| \leq 1, \quad u \in E.$$

In this case, the coorresponding result to Lemma 2.1 reads

LEMMA 4.1. Under hypothesis a), assume that there are constants $\varepsilon > 0$ and $\theta < 1$ such that the condition (4.1)-(4.2) holds. Then there exists a sequence $\{u_n\} \subset B_R$ such that

(4.3) $$F(u_n) \longrightarrow c$$

(4.4) $$F'(u_n) \longrightarrow 0 \text{ strongly in } H.$$

The proof is similar to those of the deformation theorem [11].

§ 5. The Mountain Pass Alternative.

For the function $F \in C^1(B_R,\mathbb{R})$ satisfying hypothesis a), we introduce the notations

$$\beta(u) = \frac{(F'(u),u)}{\|u\|^2} \quad \text{and} \quad \Pi(u) = F'(u) - \beta(u)u, \quad \text{for} \quad u \neq 0.$$

and suppose henceforth that the following condition holds:

b). F' is bounded below on S_R, namely, there is a constant $M > 0$ such that

$$-(F'(u),u) \leq M \quad \text{if} \quad \|u\| = R.$$

where $\|\cdot\|$ stands for the norm in the Hilbert space H.

VARIANT I. Under assumptions a) - b), the following alternative holds: Either

(I.a) There is a sequence $\{u_n\} \subset B_R$ such that

(5.1) $$F(u_n) \longrightarrow c,$$

(5.2) $$F'(u_n) \longrightarrow 0,$$

or

(I.b) There is a sequence $\{u_n\} \subset S_R$ such that (5.1) holds and

(5.3) $\Pi(u_n) \longrightarrow 0$ strongly in H

(5.4) $\limsup\, (F'u_n, u_n) < o.$

PROOF. We show how Lemma 4.1 implies Variant I. By b) the sequence $\{\beta(u_n)\}$ is bounded by and so it contains a subsequence converging to some $\beta \leq o$. As $F'(u_n) = \Pi(u_n) + \beta_n u_n$, it suffices to find a subsequence satisfying (5.3) and (5.4).

Asssume that there no sequence satisfying obtion (I.b). Then there would be constants $\varepsilon, b > o$ such that

$$\| \Pi(u_n) \| \geq b$$

for all $u \in S_R$ satisfying (4.2) and $(F'(u),u) \leq o$. Choosing $\theta < 1$ be such that $0 < \theta^{-2} - 1 < \frac{R^2 b^2}{M^2}$ we have, for all $u \in S_R$ satisfying (4.2),

$$\theta^{-2} \nu(u)^2 < R^2 \|\Pi(u)\|^2 + (F'(u),u)^2 = R^2 \|F'(u)\|^2$$

since $\|F(u)\|^2 = \|\Pi(u)\|^2 + \|u\|^{-2}(F'(u),u)^2$. This contradicts (4.1). Moreover, inequality (4.1) holds trivially when $(F'(u),u) > o$.

Now, the option (I.a) follows from Lema 4.1.

Finally, suppose that there is no sequence satisfying option (I.b) but there one if we replace (5.4) with $(F'(u_n),u_n) \longrightarrow o$. In this case $\beta(u_n) \longrightarrow o$ and (5.3) implies (5.2). Thus option (I.a) holds in this case as well. ■

We regard the Variant I.1 as a MOUNTAIN PASS ALTERNATIVE.

Next we add the following monotonicity type hypothesis:

c). If $\{u_n\} \subset S_R$ is a sequence satisfying

(5.5) $F(u_n) \longrightarrow c,$

(5.6) $\Pi(u_n) \longrightarrow 0,$

(5.7) $\limsup\, (F'(u_n),u_n) \leq o,$

then $\{u_n\}$ has a convergent subsequence.

VARIANT II. Under hypotheses a) - c) the following alternative holds: Either

(II.a) There is a solution $u \in B_R$ of

(5.8) $$F'(u) = 0, \quad F(u) = c,$$

or

(II.b) There is a solution of

(5.9) $$F'(u) = \beta u, \quad \beta < o, \quad \|u\| = R, \quad F(u) = c.$$

PROOF. Consider succesively the two alternative of Variant I.

Option (I.a) implies that $(F'(u_n),u_n) \longrightarrow o$ and (5.5)-(5.7) hold. By hypothesis c) there is a convergent subsequence whose limit will satisfy (5.8).

If option (I.b) holds then (5.5) - (5.7) are satisfied and there is a subsequence converging strongly to some $u \in S_R$ and because

$F(u_n) \to F(u)$, $F'(u_n) \to F'(u)$, $(F'(u_n),u_n) \to (F'(u),u)$, $\beta(u_n) \to \beta(u)$,

and so u is a solution of problem (5.9).■

This last variant is also called the MOUNTAIN CLIFF THEOREM.

On a heuristic level,the theorem states that if the border is impenetrable then either one walks on the same level surface or one glides off the mountain along a normal direction.

A similar framework has been considered in [9].

Until now R was a fixed constant; it will obey further to some restrictions.

VARIANT III. If hypothese a)-c) are satisfied for each $R > \|e\|$, then the following alternative holds:

(III.a) There exists a solution $u \neq 0$ of $F'(u) = 0$ or

(III.b) For each $R > \|e\|$ there exists a solution of

(5.10) $$F'(u) = \beta u, \quad \beta < o, \quad \|u\| = R, \quad F(u) = c_R.$$

Proof. We apply Variant II for each R. Note that a solution (5.4) cannot vanish by hypothesis b). ■

Finally, we add the last hypothesis:

d). $F : H \longmapsto \mathbb{R}$ is weakly lower semicontinous and $F(e) \leq F(0)$.

VARIANT IV. Under hypothese a) - d), the following alternative holds.

Either

(IV.a) there exist at least two solutions of

(5.11) $$F'(u) = 0, \quad u \neq 0$$

or

(IV.b) for each $R > \|e\|$ there exists at least one solution of

(5.12) $$F'(u) = \beta u, \quad \beta < o, \quad \|u\| = R.$$

PROOF.Assume that there is an $R > \|e\|$ for which (5.12) has no solutions. Then by Variant II, (4.8) has a solution $u \neq 0$. Let

$$m = \inf F(u) \mid u \in B_R \}$$

By hypothesis d) there is $0 \neq v \in B_R$ such that $F(v) = m$ and v will be either a solution of (5.11) or (5.12).Since (5.12) has no solution, v must satisfy (5.11). Since,

(5.13) $$m \leq F(e) < \alpha \leq c_R,$$

and so $F(v) \neq c_R$. Hence there exist two nonzero solutions of (5.11). ■

VARIANT V. Under the same hypotheses, the following alternative holds: Either

(V.a) There exist at least one solution of (5.11) or

(V.b) for each $R > \|e\|$ there exist at least two solutions of (5.12).

PROOF. Suppose that there is no solution of (5.11). Then, by Variant III, for each $R > \|e\|$ there exists a solution of (5.10). Moreover, let $v \neq 0$ the minimizer of F on B_R. This v cannot be in interior of B_R, for then it would be a solution of (5.11).It must be a solution of (5.12). It cannot satisfy $F(v) = c_R$. Hence (5.12) must have at least two solutons.■

§ 6. SEMILINEAR EQUATIONS.

We turn back to semilinear equations. Let Ω be a bounded domain in $\mathbb{R}^N$ and consider the Dirichlet problem for the operator equation

(6.1) $$Au = g(x,u)$$

where A is a selfadjoint operator on $L^2(\Omega)$, such that $(Au,u) \geq c_1^2 \|u\|_2^2$ and

$$C_o^\infty(\omega) \subset H = D(A^{1/2}) \subset H_o^m(\Omega), \quad m > o.$$

Suppose that $g:\Omega \times \mathbb{R} \longmapsto \mathbb{R}$ is a Caratheodory function satisfying the growth condition

(6.2) $$|g(x,s)| \leq c_2|s|^{p-1} + b(x), \quad b \in L^{p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1$$

and $1 \leq p < \infty$ if $N = 2$ and $1 \leq p \leq \frac{2N}{N-2}$ if $N \geq 3$. Define

$$G(x,u) = \int_0^u g(x,s)\, ds$$

and obtain that

$$|G(x,s)| \leq c_3|s|^p + a(x), \quad a \in L^1(\Omega).$$

Consider the functional

$$F(u) = \frac{1}{2}(Au,u) - \int_\Omega G(x,u)\, dx.$$

which is continuously Fréchet differentiable in $L^p(\Omega)$, (see [6]).

We introduce now the energetic space H with norm $\|u\| = \|A^{1/2}u\|_2$. Then $H \subseteq L^2(\Omega)$, $\|u\|_2^2 \leq \frac{1}{c^2}\|u\|^2$, and we have

1) There is a dense subset X in H such that

$v_n \longrightarrow v$ weakly in H implies $F'(v_n)z \longrightarrow F'(v)z$ $\forall\, z \in X$;

2) There is an $\varepsilon > 0$ such that $\nu(u) - \varepsilon\|u\|^2$ is weakly l.s.c.

Indeed, take $X = C_0^\infty(\Omega)$. If $v_n \longrightarrow v$ weakly in H then $v_n \longrightarrow v$ in $L^p(\Omega)$ and passing eventually to a subsequence a.e. in Ω. Therefore

$$\int_\Omega g(x,v_n)z\,dx \longrightarrow \int_\Omega g(x,v)z\,dx$$

and 1) is proved. Now, we note $\int_\Omega g(x,u)u\, dx$ is weakly continuous in H and

$$(F'(u),u) - \varepsilon\|u\|^2 = (1-\varepsilon)\|u\|^2 - \int_\Omega g(x,u)u\, dx$$

is weakly l.s.c. on $H \cap S_R$. Then 2) hold. ■

Hypotheses b) and c) in § 5 can be shown as a result of assumptions 1)-2).

Similarly, we have

$$\int_\Omega |G(x,u)|\,dx \leq c_4\|u\|^p + c_5$$

and the above arguments prove $\int_\Omega G(x,u_n)\, dx \longrightarrow \int_\Omega G(x,u)\, dx$. From this, hypothesis d) easily follows.

On the other hand, solutions in H of $F'(u) = 0$ are solutions of (6.1). Moreover, a solution $u \in H$ of

$$F'(u) = \beta u, \qquad \beta < 0$$

is also a solution of

$$Au = \lambda\, g(x,u) \text{ with } 0 < \lambda = \frac{1}{1-\beta} < 1.$$

THEOREM 6.1. Under hypotesis (6.2), the following alternative holds: Either

(a) there exists a solution of

(I) $$Au = g(t,x), \qquad u \in H,$$

or

(b) for each $R > 0$ there exists at least one solution of

(II) $$Au = \lambda g(t,x), \quad u \in H, \quad (Au,u) = R, \quad 0 < \lambda < 1.$$

Proof. For each $R > 0$, let $m_R = \{\inf F \mid B_R\}$. Since F is weakly l.s.c., there is a $u \in B_R$ such that $F(u) = m_R$ which will be a solution of one of the above problem. ∎

If we add now the hypothesis

There is $0 \neq u_0 \in D$ such that

(6.3) $$(Au_0,u_0) < 2\int_\Omega G(u_0,x)\,dx$$

and we have

THEOREM 5.2. In hypotheses (6.2)-(6.3), the following alternative holds: Either:

(c) there exist at least two nonzero solutions of (I) or

(d) for each $R > (Au_0,u_0)$ there exists a solution of (II).

PROOF. By (6.3), we have $F(u_0) < 0 = F(0)$. Thus there is a $\delta > 0$ such hat

$$F(u_0) < \rho = \inf_{\|u\|=\delta} F(u)$$

with $\|u_0\| > \delta$. We apply then Variant IV. ∎

Some of these results, concerning the number of solutions of semilinear equations, are known (e.g.[3]) but here they are restablished using a different approach.

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