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Classroom Notes

EULER'S METHOD FOR A FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

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Several authors have discussed the approximation of solutions of ordinary differential equation $y' = f(x,y)$, $y(0) = y_0$ by Euler's method [2].

In this paper we study, using the characteristic curves, the first order partial differential equation of the form

$$\frac{\partial u}{\partial t} + f(t,x)\frac{\partial u}{\partial x} = F(t,x,u(t,x)), \tag{E}$$

with an initial condition of the form

$$u(0,x) = \varphi(x), \quad 0 \leq x \leq 1. \tag{IC}$$

A numerical procedure for the solution of the above partial differential equation, based on Euler's method, will be discussed. Such equations occur for example in mathematical physics as describing the motion of disturbances in certain materials. Moreover, they occur in connection with higher order partial differential equations as characteristic equations [1].

The characteristic curves are determined by the system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x), \quad \frac{du}{dt} = F(t,x,u), \tag{1}$$

or

$$\frac{dx}{dt} = f(t,x), \quad x(0) = x_0, \quad 0 \leq x_0 \leq 1 \tag{2}$$

$$\frac{du}{dt} = F(t,x(t), \ u(t,x(t))), \quad u(0,x_0) = \varphi(x_0), \quad 0 \leq x \leq 1, \tag{3}$$

where du/dt is the total derivative of $u = u(t,x(t))$.

Now, let us consider the initial value problem (2), for all t, $0 \leq t \leq \ell$, $\ell > 0$. The numerical solution by Euler's method at the grid points, $x_k \simeq x(t_k)$, $k = 1,2,\cdots,n$ is given by

$$x_{k+1} = x_k + hf(t_k,x_k), \quad k = 0,1,2,\cdots,n-1, \tag{4}$$

where

$$h = \frac{\ell}{n} = t_j - t_{j-1} \text{ for } j = 0, 1, \cdots, n, \text{ and } 0 = t_0 < t_1 < t_2 < \cdots < t_n = \ell.$$

It is well known that

(5) $$\epsilon_k = |x_k - x(t_k)| = O(h) \text{ as } h \to 0 \text{ (or } n \to \infty).$$

On the other hand, the exact solution of (E),(IC) is determined from (3), assuming $x(t)$ already known. Since we do not know (generally) the exact solution $x(t)$ of (2), by means of which we can obtain $u(t, x(t)) = U(t)$ from (3), we are conducted to the approximate equations (a second application of Euler's method)

(6) $$U_{m+1} = U_m + hF(t_m, x_m, U_m), \quad m = 0, 1, 2, \cdots, u-1,$$

where $U_0 = U(0) = u(0, x_0) = \varphi(x_0)$.

The values for U_m provided by (6) must be compared with those given by

(7) $$\tilde{U}_{m+1} = \tilde{U}_m + hF(t_m, x(t_m), \tilde{U}_m), \quad m = 0, 1, 2, \cdots, u-1,$$

which follow directly from the equation (4), and $\tilde{U}_0 = U_0$.

Assume now that $F(t, x, u)$ satisfies a Lipschitz condition with respect to the second and third argument, i.e.,

(8) $$|F(t, x, u) - F(t, \bar{x}, \bar{u})| \leq L(|x - \bar{x}| + |u - \bar{u}|).$$

This condition must be valid when $(t, x), (t, \bar{x})$ are in the domain filled in by the trajectories of (2), and $u, \bar{u}$ are arbitrary.

Under condition (8), it is known that the scheme described by (7) provides good approximations for $u(t, x(t)) = U(t)$:

(9) $$\tilde{e}_k = |U(t_k) - \tilde{U}_k| = O(h), \text{ as } h \to 0.$$

The problem arising now is whether the errors occurring after the double application of Euler's method

(10) $$e_k \overset{\bullet}{=} |U(t_k) - U_k|, \quad k = 0, 1, \cdots, n,$$

are satisfying

(11) $$e_k = O(h), \text{ as } h \to 0.$$

As pointed out above, U_k are obtained easily from (6), unlike $\tilde{U}_k$ whose calculation depends on the knowledge of the solution of (2). Since

(12) $$e_k \leq \tilde{e}_k + |\tilde{U}_k - U_k|, \quad k = 0,1,2,\cdots,n,$$

and $\tilde{e}_k$ satisfies (9), it is obvious from (12) that the only fact we need to show to obtain (11) is

(13) $$|\tilde{U}_k - U_k| = O(h), \text{ as } h \to 0.$$

From (6) and (7) one derives

(14) $$\tilde{U}_{m+1} - U_{m+1} = \tilde{U}_m - U_m + h[F(t_m, x(t_m), \tilde{U}_m) - F(t_m, x_m, U_m)],$$

which, on behalf of (8) leads to

(15) $$|\tilde{U}_{m+1} - U_{m+1}| \leq (1+hL)|\tilde{U}_m - U_m| + hL|x(t_m) - x_m|,$$

with the initial condition $\tilde{U}_0 - U_0 = 0$.

According to (5), one can find $M > 0$ such that

(16) $$|x(t_m) - x_m| \leq Mh, \quad m = 0,1,2,\cdots,m \ .$$

Consequently, one has from (15):

(17) $$|\tilde{U}_{m+1} - U_{m+1}| \leq (1+hL)|\tilde{U}_m - U_m| + h^2 LM.$$

It is well known [3, p.59] that (17) implies

(18) $$|\tilde{U}_m - U_m| \leq M(e^{L\ell} - 1)h,$$

which proves the validity of (13).

Summarizing the above discussion, we can state the following:

Theorem. The Euler's (polygonal) method for the numerical solution of the equation (E), under condition (IC), as described above, is convergent if $f(t,x)$ and $F(t,x,u)$ are continuous and satisfy a Lipschitz condition with respect to x, resp. (x,u). The error is of the same order as h.

Remark. Using a similar approach, higher order (in h) procedures can be applied with improved precision.

References:

[1] F. John, Partial differential equations, (2nd ed.) Springer-Verlag, 1975.

[2] D. Greenspan, Discrete Numerical Methods in Physics and Engineering, Vol. 107, Academic Press, Inc. 1974.

[3] L. Collatz, The Numerical Treatment of Differential Equations (3rd Ed.), Springer-Verlag, 1959.