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Classroom Notes

TWO EXAMPLES IN LQ-OPTIMAL CONTROL THEORY

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In this paper, two examples in LQ-optimal control theory are discussed. The first example is concerned with an optimal control problem whose input-output equation is $x=Ax+Bu$ with the spectral radius of A, $r_A<1$. An application of this example to integral equations is given. The second example is concerned with the infinite horizon problem with partial initial conditions. This second example exploits admisibility hypotheses.

1. THE INPUT-OUTPUT EQUATION $x=Ax+Bu$ WITH $r_A<1$

Consider the optimal control problem described by the abstract system

(1.1) $$x=Ax + Bu$$

and cost functional of the form

(1.2) $$\mathfrak{C}(x,u) = \langle Px,x\rangle+\langle Qu,u\rangle$$

where A and B are abstract operators ; $A \in LB(X,X)$, X is a real Hilbert space (of state variables) or a real Banach space which is isomorphic to a Hilbert space (see Zhu [11]) while $B:U \subset Y \to X$, $B \in LB(Y,X)$, Y is a real Hilbert space of controls and U a closed convex set in Y. P and Q are self-adjoint bounded operators ; $P: X\to X$ is positive semi-definite while $Q:Y\to Y$ is positive definite.

Theorem 1.1: Consider the abstract system (1.1) and the cost functional (1.2) and suppose the following assumptions hold true:

(i) $A:X\to X$ is a linear continuous operator;

(ii) $B:U \subset Y\to X$ is a linear continuous operator from the closed convex set U of a real Hilbert space Y into X, $B \in LB(U,X)$;

(iii) The spectral radius of A satisfies $r_A<1$;

(iv) $\langle Qu,u\rangle \geq \lambda||u||^2_Y$, $\lambda>0$, $Q:Y\to Y$, and $P:X\to X$, $\langle Px,x\rangle \geq 0$.

Under these assumptions , there exists a unique $\tilde{u} \in U$ such that

$$\mathfrak{C}(\tilde{x},\tilde{u}) = \min \mathfrak{C}(x,u),\ u \in U$$

where

$$\tilde{x} = A\tilde{x} + B\tilde{u} .$$

Proof: Equation (1.1) can be written in the form

$$(I-A)x=Bu.$$

Since $r_A<1$, $(I-A)^{-1}$ exists and is bounded, (see Kolmogorov and Fomin [7]). Hence we can write (1.1) in the form

$$x= (I-A)^{-1}Bu = Tu, \tag{1.3}$$

where $T= (I-A)^{-1}B$. Because B is a linear bounded operator, T is a bounded operator. Equation (1.3) is input-output equation.

The abstract LQ-optimal control problem is the following: find $\tilde{u} \in U$ such that

$$\min \{<Px,x>+<Qu,u>; x=Tu\} \tag{1.4}$$

is attained at $\tilde{u}$. By a result of Corduneanu [6], under the assumptions considered in Theorem 1.1 on X,Y, P,Q, A, and B, there exists a unique $\tilde{u} \in U$ providing the solution in (1.4).

Examples and applications

(a) Integral Equations in L^2- space:

Consider the abstract system (1.1) where x belongs to the state space $L^2([0,T],R^n)$ and u belongs to the control space $L^2([0,T], R^m)$. Let A and B be Volterra integral operators:

$$(Ax)(t) = \int_0^t k(t,s)x(s)ds, \quad (Bu)(t) = \int_0^t h(t,s)u(s)ds. \tag{1.5}$$

Equation (1.1) can be written in the form

$$x(t) = (Ax)(t) + (Bu)(t). \tag{1.6}$$

with

$$A: L^2([0,T],R^n) \rightarrow L^2([0,T],R^n),$$
$$B:U \subset L^2([0,T],R^m) \rightarrow L^2([0,T],R^n).$$

$k \in L^2(\Delta,\mathcal{L}(R^n,R^n))$ while $h \in L^2(\Delta,\mathcal{L}(R^m, R^m))$ where $\Delta=\{(t,s):0 \leq s \leq t \leq T<\infty\}$.
Let $X= L^2([0,T],R^n)$ and $Y= L^2([0,T],R^m)$. Then $A \in LB(X,X)$ and $B \in LB(U,X)$ where U is a closed convex set in Y.

The solution of equation (1.6) can be written in the form (See Tricomi [10] , Corduneanu [5])

$$x(t) = \int_0^t \tilde{k}(t,s)(Bu)(s)ds + (Bu)(t) = (Tu)(t) \tag{1.7}$$

with $\tilde{k} \in L^2(\Delta,\mathcal{L}(R^n,R^n))$ and Bu an element of $L^2([0,T],R^n)$. $\tilde{k}(t,s)$ is called the resolvent kernel associated with k(t,s). Since A is a Volterra operator , $r_A=0$ hence $r_A<1$ is automatically satisfied and $(I-A)^{-1}$ is a bounded operator. Equation (1.7) is the input-output equation.

Attached to equation (1.7) is the cost functional .

$$(1.8)\qquad \mathcal{C}(x,u) = \int_0^T \{<(Px)(t),x(t)> + <(Qu)(t),u(t)>\}dt.$$

P is a self-adjoint positve semi-definite bounded operator defined on $L^2([0,T],R^n)$ while Q is a self-adjoint positive definite bounded operator defined on $L^2([0,T],R^m)$. A,B, $L^2([0,T],R^n)$, $L^2([0,T],R^m)$, U, P, and Q satisfy the hypotheses of Theorem 1.1. Hence there exists a unique $\tilde{u} \in U$ such that min $\mathcal{C}(x,u)$ is attained at $\tilde{u} \in U$.

2. THE INFINITE HORIZON PROBLEM WITH PARTIAL INITIAL CONDITIONS

Admisibility hypotheses and representation of solution of the differential system:

Consider the system (2.1)

$$(2.1)\qquad \dot{x}(t) = A(t)x(t) + f(t),\ t \geq 0,$$

where A(t) is an $n \times n$ matrix valued function, $A(t) \in \mathcal{M}(R_+,R^n)$, and $f \in L^2(R_+,R^n)$. Let P_0 be the projection of R^n onto the subspace X_0 which consists of the values for t=0 of bounded solutions of

$$(2.2)\qquad \dot{x}(t) = A(t)x(t).$$

Suppose X(t) is the fundamental matrix of the homogeneous system (2.2) with $X(0) = I_{n\times n}$ and satisfies the matrix differential equation

$$(2.3)\qquad \dot{X}(t) = A(t)X(t).$$

Solution of (2.1), with the partial initial condition

$$(2.4)\qquad P_0x(0) = \xi_0 \in X_0,$$

is given by (Massera and Schäffer [8], Coppel [2])

$$(2.5)\qquad x(t) = X(t)\xi_0 + \int_0^t X(t)P_0X^{-1}(s)\, f(s)ds - \int_t^\infty X(t)P_1X^{-1}(s)f(s)ds,$$

where $P_1 = I-P_0$.

The following theorem which is adapted from a theorem due to Conti [1] establishes the necessary and sufficient condition for (L^2,L^∞) to be an admissible pair of spaces with respect to (2.1).

Theorem 2.1: Equation (2.1), (2.4) has at least one bounded solution for every $f \in L^2(R_+,R^n)$ if and only if there is a constant K>0 such that for any $t \geq 0$,

$$(2.6)\qquad \int_0^t |X(t)P_0X^{-1}(s)|^2ds + \int_t^\infty |X(t)P_1X^{-1}(s)|^2ds \leq K^2.$$

For the definition and details of the admisibility concept used in this paper, the reader is referred to the classic book of Massera and Schäffer [9].

Let us also mention that the map $(\xi_0,f) \rightarrow x$ is continuous from $X_0 \times L^2(R_+,R^n)$ into $L^\infty(R_+,R^n)$.

Let us now apply admissibility hypotheses established above to the following LQ-optimal control problem.

Consider the LQ-optimal control problem described by the differential system

(2.7) $$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ t \geq 0,$$

and cost functional of the form

(2.8) $$\mathcal{C}(x,u) = \int_0^\infty \{<(P(t)x(t),x(t)> + <(Q(t)u(t),u(t)>\}dt,$$

where x in the cost functional are only those which satisfy $P_0x(0) = \xi_0 \in X_0$. B(t) is an $n \times m$ matrix valued function whose elements are essentially bounded on R_+. P(t) is a positive semi-definite matrix valued function while Q(t) is a positive definite matrix valued function. Furthermore,

(2.9) $$\int_0^\infty |P(t)|dt = ||P||_{L^1} < \infty, \qquad \Big(\int_0^\infty |Q(t)|^2 dt\Big)^{\frac{1}{2}} = ||Q||_{L^2} < \infty.$$

We also have $1 \leq m \leq n$, where $L^2(R_+, R^m)$ is the control space.

It is essential to observe that choosing the initial condition $P_0x(0) = \theta \in X_0$ does not diminish the generality of the problem. Indeed, by the substitution

$$x(t) = X(t)\xi_0 + y(t)$$

one obtains

$$\dot{x}(t) = \dot{X}(t)\xi_0 + \dot{y}(t) = A(t)X(t)\xi_0 + A(t)y(t) + B(t)u(t),$$

so

$$\dot{y}(t) = A(t)y(t) + B(t)u(t), \text{ with } P_0y(0) = \theta \in X_0.$$

Therefore without loss of generality we can deal with the problem of minimizing $\mathcal{C}(x,u)$ subject to (2.7) and the initial condition

(2.10) $$P_0x(0) = \theta \in X_0.$$

(2.7), (2.10) is equivalent to

(2.11) $$x(t) = \int_0^t X(t)P_0X^{-1}(s)B(s)u(s)ds - \int_t^\infty X(t)P_1X^{-1}(s)B(s)u(s)ds = (Tu)(t).$$

(2.11) is the input – output equation.

Existence and uniqueness of the optimal control

Theorem 2.2: Consider the optimal control problem described by (2.7), (2.10), and (2.8) with $\min\mathcal{C}(x,u)$ to be found. Assume that the following hypotheses hold true:

(i) $A(t) \in \mathcal{M}b(R_+, R^n)$ and B(t) is an $n \times m$ matrix valued function whose elements are essentially bounded on R_+;

(ii) The hypotheses of Theorem 2.1 hold true, in particular, inequality (2.6) is satisfied;

(iii) P(t) and Q(t) are matrix valued functions, P(t) being positive semi-definite and Q(t) positive and verify condition (2.9);

(iv) The control u is restricted to a closed convex set $U \subset L^2(R_+, R^m)$.

Then, there exists a unique control $\tilde{u} \in U$ such that

$$\mathcal{C}(\tilde{x}, \tilde{u}) = min\mathcal{C}(x,u), \ u \in U$$

where

$$\tilde{x} = T\tilde{u}.$$

Remark 2.1: Observe that $x \in \mathcal{H}$, the space of those locally integrable functions $x: R_+ \to R^n$ with the scalar product given by

$$<x,y>_P = \int_0^\infty <P(t)x(t),y(t)>dt<\infty$$

while the norm is given by

$$||x||_P = \left(\int_0^\infty <P(t)x(t),x(t)>dt\right)^{\frac{1}{2}}.$$

We shall represent $\mathcal{H}$ by $L_P^2(R_+, R^n)$. Clearly $L^\infty \subset L_P^2(R_+, R^n)$.

Proof: Let $u \in U \subset L^2(R_+, R^m)$ be closed and convex. The optimal control problem is to determine the control $\tilde{u} \in U$ such

that

(2.12) $$\mathcal{C}(\tilde{x}, \tilde{u}) = \min\mathcal{C}(x,u), \ u \in U$$

where $\tilde{x} = x(t;\tilde{u})$ is the optimal trajectory, determined by (2.7), (2.10). The existence of bounded solutions of (2.7), (2.10) on R_+ is a consequence of admisibility hypotheses. In order to prove the existence and uniqueness of $\tilde{u}$, let us introduce another scalar product $<<.,.>>$ in the space $L^2(R_+, R^m)$ such that its topology is preserved. Define $<<.,.>>$ by

(2.13) $$<<u,v>> = \int_0^\infty \{<(P(t)x(t),y(t)> + <(Q(t)u(t),v(t)>\}dt,$$

where x=Tu and y=Tv. $<<.,.>>$ is clearly a scalar product. Observe that $<<u,u>> = 0$ implies $<<Px,x>>=0$, $<<Qu,u>>=0$ a.e. on R_+. Since P(t) is positive semi-definite and Q(t) is positive definite, this will only hold if u is identically $\theta \in L^2(R_+, R^m)$. We proceed now to show that $<<.,.>>$

is equivalent to the L^2-norm.

Since $<P(t)x(t),x(t)> \geq 0$, and $<Q(t)u(t),u(t)> \geq \alpha||u||^2_{L^2}$ with $\alpha>0$, the following inequalities are valid:

(2.14) $$\alpha||u||^2_{L^2} \leq \mathcal{C}(x,u) = <<u,u>> \leq \beta||u||^2_{L^2}$$

where

(2.15) $$\beta = (||P||_{L^1}\, ||T||^2 + ||Q||_{L^2}),$$

and

$$||u||_{L^2} = \left(\int_0^\infty ||u(t)|^2 dt\right)^{\frac{1}{2}}.$$

Inequalities (2.14) show equivalence of the two norms. The new norm is $\{\mathcal{C}(x,u)\}^{\frac{1}{2}} = \{<<u,u>>\}^{\frac{1}{2}}$. Hence the optimal control problem is reduced to a well known result which states the following: every closed and convex set in a Hilbert space has an element of minimum norm. Since U remains closed and convex in the new topology, the optimal control $\tilde{u} \in U$ exists and $\tilde{x} = T\tilde{u}$, is unique.

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