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Classroom Notes

PERIODIC TRAVELLING WAVES IN NONLINEAR DIFFUSION MODELS

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The one-dimensional nonlinear diffusion model is provided by the equation

$$-u_t + u_{xx} + f(u, u_x) = 0, \tag{E}$$

where $u = u(t,x)$ is sought in the class $\mathcal{C}^{(2)}$ and whose domain of definition is the whole plane (t,x), or at least the half-plane $t \geq 0, -\infty < x < \infty$.

A travelling wave associated with (E) is a solution of (E) having the special form

$$u = y(x - at), \quad a = \text{const}. \tag{S}$$

A periodic travelling wave will be a solution of the form (S) which corresponds to a periodic y. If such solutions do exist, then they will be periodic in t and x, as seen from (S).

From (S) we obtain $u_t = -ay'$ and $u_x = y'$, $u_{xx} = y''$. Therefore, a solution of the form (S) must satisfy the second order ordinary differential equation

$$y'' + ay' + f(y, y') = 0. \tag{1}$$

Let us consider now the special form of $f(y,z)$ given by

$$f(y,z) = f_0(y)z + g(y), \tag{2}$$

which means that we look at the case when $f(y,z)$ is linear in z.

The equation (1) becomes

$$y'' + f(y)y' + g(y) = 0, \tag{3}$$

with

$$f(y) = a + f_0(y). \tag{4}$$

It is well known that equation (3), which is usually called the Liénard equation, possesses periodic solutions under rather mild conditions. These conditions regard only the functions $f(y)$ and $g(y)$, with $g(y)$ given by (4). Since (4) involves the constant a, which is not assigned, there remains to impose adequate conditions on $f_0(y)$ to ensure the existence of periodic solutions to (3)..

As we know, the existence of non-trivial periodic solutions to (3) is investigated by looking for limit cycles to the plane system

(5) $$y' = z - F(y), \quad z' = -g(y),$$

where $F(u) = \int_0^u f(y)dy$.

The following conditions will ensure the existence and uniqueness of a limit cycle to (5), which means (3) has a unique period solution which does not reduce to a constant [2], [4], [5], [6]:

1) $f: R \to R$ is continuous, even, $f(0) < 0$, and $g: R \to R$ is locally Lipschitz continuous and odd;
2) $yg(y) > 0$ for $y \neq 0$;
3) $F(u)$ has a single positive zero $u = b$, and $F(u) \uparrow \infty$ for $u > b$.

Moreover, the unique limit cycle is orbitally stable, which means that the trajectories of (5) spiral towards the closed trajectory if they pass through points in a neighborhood of this trajectory. All the spiraling trajectories correspond to asymptotic periodic solutions of Liénard's equation (3). Of course, all these solutions of (3) generate travelling waves for the model described by (E), with $f(y,z)$ given by (2).

A special attention must be paid to those conditions regarding the function $f(y)$, as defined by (4). Since it involves the wave speed a, which is not assigned in advance, we can expect that in terms of $f_0(y)$ we will be able to determine an admissible range for this variable.

Indeed, in accordance with condition 1) formulated above, the following inequality must be verified.

$$a < f_0(0). \tag{6}$$

Besides (6), one must also satisfy the requirements of condition 3). The function $f_0(y)$, which we assume to be continuous and even on R, and the wave speed a must be such that

$$F(y) = ay + \int_0^y f_0(u)du, \quad y \in R, \tag{7}$$

has a single positive zero b, and it is monotonically increasing at ∞ with y.

The monotonicity condition means

$$a + f_0(y) \geq 0 \text{ for } y > b, \tag{8}$$

while $F(y)\uparrow\infty$ means

$$\lim_{y\to\infty}\left[ay + \int_0^y f_0(u)du\right] = \infty. \tag{9}$$

Conditions (6), (8) and (9) show the interrelations between a and $f_0(y)$. Since $f_0(y)$ must be an even function (in order to assure the same property for $f(y)$ given by (4)), it remains to use (6), (8) and (9) to determine the interval in which the wave speed a can be chosen, given $f_0(y)$.

For instance, if we take $f_0(y) = 3y^2 - 1$, then (6) becomes $a < 1$. Since $F(y) = y^3 - (1-a)y$, we have $b = \sqrt{1-a}$ and both conditions (8) and (9) are obviously satisfied, b being the only positive zero of $F(y)$. Consequently, for $f_0(y) = 3y^2 - 1$, the only restriction we must impose to a is $a < 1$. A similar situation appears when we choose $f_0(y) = 2|y| - 1$.

In conclusion, the system (5) has a unique limit cycle under the above mentioned conditions. The only singular point of (5) being (0,0), which must be inside the cycle, there results that any trajectory of (5) passing through a point other than the origin must spiral towards the cycle - in one sense - and towards the origin in the other sense.

In other words, besides the unique periodic solution $y = y(x - at)$ of (3), which corresponds to the limit cycle, there exist infinitely many solutions which generate travelling waves for (E), with $f(y,z)$ given by (2), and whose behavior at infinity is asymptotically periodic (in one sense).

The trajectories outside the limit cycle will also spiral - in one sense only - towards the cycle.

Of course, we cannot state any result concerning the behavior of other solutions of (E), which are not generating travelling waves of the form (S). If we look at the model of diffusion described by

$$-u_t + u_{xx} + f(u) = 0, \tag{10}$$

which has been thoroughly investigated in the literature [1],[3],[5],[6],[7], then we are tempted to conjecture the fact that the solutions of (E), under assumption (2), do converge as $t \to \infty$ to one of the solutions represented by a travelling wave (or a translated of such solution, say $y(x - x_0 - at)$, because this is also a solution of (3)). The behavior of travelling waves as $t \to -\infty$ is described under adequate conditions, by the convergence to zero. The major question is whether this behavior is valid for solutions of (E), under convenient assumptions on initial data. To the best of our knowledge, this question did not get an answer in the literature.

On the other hand, the approach based on the theory of limit cycles for plane systems leads to other interesting illustrations of the kind of behavior appearing in the diffusion model described by the equation (E). Among them, let us mention a case when all trajectories of (5) in a neighborhood of $x = y = 0$ are closed: $f(x) = f(-x)$, $g(x) = g(-x)$, $xg(x) > 0$ for $x \neq 0$, and for some $a > 0$

$$\int_0^y \frac{g(x)}{F(x)} dx \geq (\tfrac{1}{4} + \epsilon) \, | F(x) | , \quad 0 \leq x \leq a, \tag{11}$$

where $\epsilon > 0$. This result is due to Z. Opial and is quoted in [6].

Many other possibilities remain to be exploited, taking into account recent results concerning the topology of limit cycles for plane autonomous system [5],[7].

The aim of this note is only to signal this approach in the investigation of travelling waves solutions in a model of the form (E).

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