# DOES THE QUADRATIC EQUATION HAVE GREEK ROOTS? A STUDY OF "GEOMETRIC ALGEBRA", "APPLICATION OF AREAS", AND RELATED PROBLEMS

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## III

1. In this section, we would like to focus attention on what we believe to be the central issue surrounding the controversy over "geometric algebra," namely the contention of those writers, who discern an algebraic substructure in the Elements and elsewhere in Greek mathematics, that the Greeks solved quadratic equations by utilizing geometry.[95] This, we feel, is the real litmus test for the historical efficacy of the "geometric algebra" concept. Otto Neugebauer once said that the problem of "application of areas" is "the central problem of the geometrical algebra."[96] This makes perfect sense once we realize that it is the "application of areas"-technique above all else that underlies the propositions in the Elements (e.g., II.11 and II.14) that deal with finding unknown magnitudes and that have been typically associated with "geometric algebra." In other words, the "application of areas" is, for Neugebauer et al., that subdivision of "geometric algebra" that treats problems allegedly equivalent to the solution of second degree equations i.e., it is the subdivision which handles that peculiarly algebraic entity--the unknown quantity. It is, therefore, only natural that our study should focus on quadratic equations; one should be aware, however, that the proponents of "geometric algebra" have found that many of the most important problems in Greek geometry amount to nothing more than the solution to some form of an algebraic equation, be it linear, quadratic, cubic, or biquadratic. As a typical instance of this sort of interpretation consider the following views of H. G. Zeuthen on Euclid, Book X:

> Da nun solche Wurzeln von Gleichungen zweiten Grades, die mit den gegebenen Grössen inkommensurabel werden, sich nicht durch diese und durch Zahlen ausdrücken lassen, so ist es begreiflich, dass die Griechen bei exakten Untersuchungen

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> keine Näherungswerte einführten. sondern weiter operierten mit den gefundenen Grössen, die dargestellt wurden durch die Strecken, die sich aus der, der Lösung der Gleichung entsprechenden, Konstruktion ergaben. Es ist das ganz ebenso, wie wenn wir Wurzeln nicht ausrechnen, sondern uns damit begnügen diese durch Quadratwurzelzeichen und andere algebraische Zeichen auszudrücken. Da indessen eine Strecke wie die andere aussieht, so erhielt man dadurch nicht denselben Überblick, den die Zeichensprache uns gewährt. Deshalb wurde es notwendig eine Klassifikation der irrationalen Grössen vorzunehmen, die sich durch successive Lösung von Gleichungen zweiten Grades ergeben hatten.[97]

After citing Tannery's concurrence with the above, Heath remarks:

> Accordingly Book X. formed a repository of results to which could be referred problems which depended on the solution of certain types of equations, quadratic and biquadratic but reducible to quadratics.[98]

It may well be that the existence of solutions to geometric problems substantially equivalent to certain algebraic equations would, in fact, constitute very strong circumstantial evidence favoring the view that there was an underlying algebra motivating Greek geometry. But, as the arguments we present in this section clearly show, the alleged correspondence between Greek geometry and elementary algebraic techniques is not at all good. And, even more importantly, we will indicate how the attempt to understand Greek mathematics as algebraically motivated leads to paradoxical conclusions that make nonsense out of what we find in the Greek texts themselves.

2. The significance of the "application areas" is attested to by several ancient sources who attribute its discovery to the Pythagoreans. The most important of these is the following testimony of Eudemus, the author of the <u>Γεωμετρικὴ ἱστορία</u>, as reported by Proclus:

> Eudemus and his school tell us that these things--the application (παραβολή) of areas, their exceeding (ὑπερβολή), and

> their falling short (ἔλλειψις)--are ancient discoveries of the Pythagorean muse. It is from these procedures that later geometers [notably Apollonius] took these terms and applied them to the so-called conic lines, calling one of them "parabola," another "hyperbola," and the third "ellipse," although those godlike men of old saw the significance of these terms in the describing of plane areas along a finite straight line.... Euclid too in his sixth book [VI.27-29] speaks in this sense of "exceeding" and "falling short"; but here [I.44] he needed "application," since he wished to apply to a given straight line an area equal to a given triangle, in order that we might be able not only to construct a parallelogram equal to a given triangle, but also to apply it to a given straight line....[99]

Further testimony, beyond this account of Eudemus, to the importance of "application of areas" can be found in Plutarch, where it is seen as holding a central place in Pythagorean lore. Now it is true that Walter Burkert has assembled strong evidence suggesting that such sophisticated mathematics had nothing to do with Pythagoreanism,[100] and indeed it certainly must be admitted that there is a strikingly legendary quality suffusing the passages from Plutarch that we are about to consider. For example:

> Now among the most characteristic theorems, or rather problems, of geometry is this: given two figures, to construct a third equal to one and similar to the other. [Cf. Elements, VI.25, which, as we shall see, plays a pivotal role in the "application of areas."] They say, in fact, that Pythagoras offered sacrifice when he solved this problem; for it is surely much more elegant and inspired than that famous theorem which gave the proof that the square on the hypotenuse is equal to the sum of the squares on the sides enclosing the right angle.[101]

Regarding this passage, Heath (following Bretschneider and Hankel) points out that the account of the sacrifice is inconsistent with Pythagorean ritual, which strictly forbade the practice, but unlike Burkert, he sees no reason to disbelieve this and other ancient accounts that attribute the discovery of "application of areas" and much other sophisticated mathematics to the Pythagoreans.[102] And, in effect, the characteristic Pythagorean mysticism comes prominently to the fore when the preceding passage is placed in its proper context, namely as part of a discussion of Pythagorean cosmology. The passage continues:

> . . . recall the threefold division, in the Timaeus, of the first principles from which the cosmos came to birth. One of them we call, by the most appropriate of names, God, one matter, and one form. Matter is the least ordered of substances, form the most beautiful of patterns, and God the best of causes. Now God's intention was, so far as possible, to leave nothing unused or unformed, but to reduce nature to a cosmos by the use of proportion and measure and number, making a unity out of all the materials which would have the quality of the form and the quantity of the matter. Therefore, having set himself this problem, these two being given, he created a third, and still creates and preserves throughout all time that which is equal to matter and similar to form, namely, the cosmos.[103]

Thus God created the cosmos by continuous application of Euclid VI.25! Certainly it would be difficult to imagine a more exalted view of a geometrical proposition than this. Another passage in Plutarch alludes to the same incident involving Pythagoras' sacrifice; but this time it is told in order to illustrate the wild extremes of divine genius. We quote the relevant passage as part of this larger context, both to illustrate the flavor of Plutarch's account and for the intrinsic interest of the legends themselves:

> . . . Eudoxus prayed to be consumed in flames like Phaëthon if he could but stand next to the sun and ascertain the shape, size, and composition of the planets, and when Pythagoras discovered his theorem he sacrificed an ox in honour of the occasion, as Apollodorus says:
>
> When for the famous proof Pythagoras
> Offered an ox in splendid sacrifice--
>
> whether it was the theorem that the square on the hypotenuse is equal to the sum of the squares on the sides of the right

> angle or a problem about the application of a given area. His servants used to drag Archimedes away from his diagrams by force to give him his rubbing down with oil; and as they rubbed him he used to draw the figures on his belly with the scraper; and at the bath, as the story goes, when he discovered from the overflow how to measure the crown, as if possessed or inspired, he leapt out shouting 'I have it' ['εὕρηκα' ] and went off saying this over and over.[104]

When our sources are embedded in such fantastic accounts they do not inspire confidence in their authenticity, and we are compelled to acknowledge that Burkert's views are not without merit. Still our purpose here is not so much to evaluate the claim that "application of areas" was a Pythagorean discovery as it is simply to document the immense importance of "application of areas" in Greek thought. For this purpose, the foregoing passages are ample enough testimony.

3. The "application of areas," as we remarked earlier, should be taken as a litmus test for assessing the historical and mathematical cogency of the entire edifice of "geometric algebra." The essential ingredients of this technique can be understood by examining two important trains of thought in the Elements that culminate with the proofs of Propositions VI.27-29. These two chains of ideas can be discerned in the following propositions, which we must study in detail:

1) I.42 - I.45 - VI.25 - VI.28 and 29.

2) I.45 - I.47 - II.5 and 6 - II.11 and 14 - VI.27-29.

Before considering the above chains of propositions, however, a few comments should be made pertaining to the remaining results in "geometric algebra," i.e., those which are relatively unrelated to "application of areas." If "application of areas" is the subdivision of "geometrical algebra" that allegedly develops techniques for solving "equations." it is only natural to ask: What about the 'purpose' of other "geometrical-algebraic" propositions like II.1-4, II.7-10, and II.12-13, that have nothing to do with equations? The answer, we are told, is that they are algebraic identities;[105] their only purpose is to demonstrate various relationships that arise in the process of transforming two-dimensional figures, particularly rectangles. It should, however, be noted that there is a grey area here, containing propositions like II.5 and 6 which are also indentities, but which we have earmarked for further study precisely because of their importance for "application of areas." Another example is provided by propositions I.47 and I.48, the "Pythagorean theorem" and its converse, which, viewed algebraically, are simply identities involving the given sides of a right triangle. Viewed as a geometric transformation,[106] I.47 states that in a right triangle the square on the hypotenuse can be transformed into the two squares on the sides forming the right angle and vice-versa. I.48, on the other hand, says that if a square can be so transformed, then the sides of the three given squares comprise three sides of a right triangle. Although I.47 and 48 both express mathematical facts that are relatively deep, compared with most of the results in Book II, from the point of view of "geometric algebra," they too are mere identities bearing only an incidental relationship to the machinery for solving equations, viz., "application of areas."

Thus, the first ten propositions in Book II of the Elements are, from the point of view of "geometric algebra," nothing but algebraic identities.[107] Furthermore, II.12 and II.13 are identities in the same sense that I.47, 48 are, since they are the geometric formulations of what today is treated as a single theorem, namely the Law of Cosines, a result that generalizes the Pythagorean theorem to the case of an arbitrary triangle (i.e., not necessarily right-angled). This leaves only two propositions in Book II, II.11 and II.14, and these form part of the nucleus of results that was allegedly used for the Euclidean solution of algebraic equations.

Now the ability to solve first-and second-degree equations is of paramount importance in elementary algebra. It is our contention, in fact, that a mathematical technique that fails to develop this far is better regarded as advanced arithmetic rather than as elementary algebra. Even for a "geometrical algebraist," the first ten propositions of Book II, for example, do not, in and of themselves, go beyond the stage of generalizing certain arithmetical relations formulated in geometric terms. None of these relations requires anything comparable to the notion of an unknown quantity, i.e., a magnitude whose value can be found via a coherent process of formal arithmetical procedures. Yet it is this very idea that, according to Freudenthal,[108] is most fundamental and basic, even characteristic, of the entire algebraic enterprise, and it is for this reason that "application of areas," the "subdiscipline" that allegedly deals with solving equations, i.e., finding unknowns, is the crucial test for the entire concept of "geometric algebra." If this assessment of the nature of algebra is correct, then there is every reason to believe that the two key chains of propositions presented above, displaying as they do the logico-mathematical underpinnings leading to the fundamental propositions having to do with "application of areas" (VI.27-29), are of the utmost importance in answering the question: "Just how algebraic is 'geometric algebra'?" It is, therefore, time that we examine them in detail. The conclusion of our investigation will show, among other things, the ahistoricity involved in seeing VI.27-29 as Greek solutions to quadratic equations.

4. We begin with the first chain of propositions, I.42 - I.45 - VI.25 - VI.28 and 29. Proposition I.42 is, in effect, a lemma required in order to prove I.44 which reads:

> To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.[109]

The lemma, I.42, differs only in that the parallelogram need not be applied to a given line, i.e., it need not have one side equal to a prescribed length:

> To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.[110]

Let us now apply this result to the proof of I.44. Thus we are given (see fig. III.1) a straight line AB, a triangle C, and a rectilineal angle D. We wish to apply a parallelogram to AB in an angle equal to D and in such a way that the parallelogram is equal (in area) to the given triangle C.

For the proof, first use I.42 to construct parallelogram BEFG equal to C with BE extending AB and with angle EBG equal to D. Next, draw parallelogram AHGB, using I.31,[111] and join HB. We then observe that angle BHG together with angle HFE is less than angle AHF together with angle HFE, whereas the latter two equal the sum of two right angles, by I.29.[112] Thus Postulate 5[113] implies that when HB and FE are produced, they will eventually intersect (say in K). We can now reapply I.31, extend GB

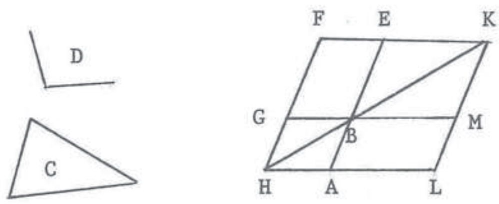



Fig. III.1

to M, and construct parallelogram HFKL. Since angle GBE is equal to angle ABM, by I.15,[114] it follows that angle ABM is equal to D. Further, I.43 states that:

> In any parallelogram the complements of the parallelo-

> grams about the diameter are equal to one another.[115]

Thus LB equals BF which equals triangle C, and therefore ABML is the desired parallelogram since it satisfies all the required conditions.

As a fairly immediate corollary to this, there is the following Proposition, I.45:

> To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.[116]

The proof involves the following ideas. First, a rectilineal[117] figure can be "triangulated," i.e., decomposed completely into triangles. It follows that if we can prove the result for an arbitrary quadrilateral (i.e., two triangles), an n-fold application of the same proof will prove the theorem in general.

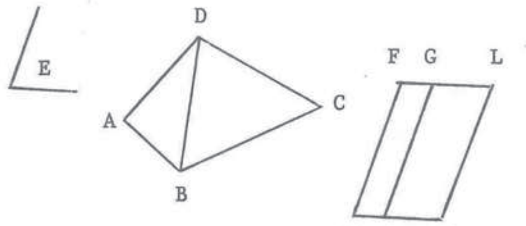


Fig. III.2

To prove the result for the quadrilateral pictured in Fig. III.2, one simply applies I.42 to obtain parallelogram KHGF equal to triangle ABD with angle FKH equal to E. Then, utilizing I.44, apply to GH a parallelogram HMLG equal to triangle BDC and with angle GHM equal to E. The remainder of the proof is a routine check that the figure KMLF so obtained satisfies the conditions of the theorem. Now, the importance of I.45 should be judged in light of the following remarks of Proclus:

> It is my opinion that this problem is what led the ancients to attempt the squaring of the circle. For if a parallelogram can be found equal to any rectilinear figure, it is worth inquiring whether it is not possible to prove that a rectilinear figure is equal to a circular area.[118]

The next step along our path is the beautiful Proposition VI.25 alluded to in the passages cited from Plutarch:

> To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal

The proof in the Elements is as follows. We are given (see Fig. III.3) two rectilineal figures ABC and D. We must construct a figure similar to the first and equal to the second. To begin with, we use I.44 to apply to the line BC parallelogram BE equal to triangle ABC (the angle of application is arbitrary). Next, we use I.45 (actually I.45A, cf. our discussion above in section I, #7) to apply parallelogram CM to CE, in such a way that CM equals D and angle FCE is equal to angle CBL.

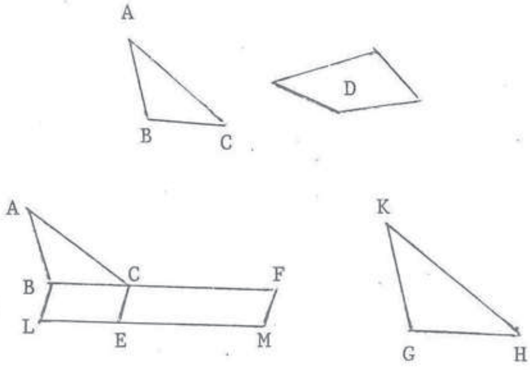



Fig. III.3

Now construct GH, the mean proportional to BC and CF by using VI.13[120] (or, we may say, II.14), and taking the newly formed line GH, construct figure KGH similar to ABC by applying VI.18.[121] Now BC:GH = GH:CF; hence, by the Porism to VI.19,[122] BC:CF = (fig. ABC): (fig. KGH). But by VI.1,[123] BC:CF = (▱BE):(▱EF), from which it follows that (ΔABC):(ΔKGH) = (▱BE):(▱EF), and as ABC is equal to ▱BE, it follows that KGH equals ▱EF.[124] Therefore, the triangle KGH, which is similar to ABC, and also equal to ▱EF = D, satisfies the conditions of the theorem.

The fact that there is no loss of generality in letting the first rectilineal figure be a triangle is not explained in the text, and, as a matter of fact, there seems to be no simple way of advancing from this special case to the proof of the general result. Probably the best way to remedy this flaw in the proof is to replace ABC by an arbitrary rectilinear figure and use I.45A rather than I.44 in order to obtain the parallelogram BE. If this minor modification is made, the rest of the proof goes through, more or less, as before.[125]

The last link in this particular chain of theorems is represented by Propositions VI.28 and 29, both of which require VI.25 as a key step in their respective proofs. These two propositions are almost identical in form, but VI.28 is complicated by the necessity of stating a διορισμός (here, a condition for possibility of solution) which is worked out beforehand in Proposition VI.27. Because VI.28 and 29 are so similar, it will be sufficient for our purposes to consider only one of them. We have decided to focus our attention on VI.29, because it requires no διορισμός and, in addition, is intimately related to an important result in Book II (Proposition

II.11). But first let us record what Heath has to say concerning the significance of these propositions:

> The importance of VI.27-29 from a historical point of view cannot be overrated. They give the geometrical equivalent of the algebraical solution of the most general form of quadratic equation when that equation has a real and positive root. It will also enable us to find a real negative root of a quadratic equation; for such an equation can, by altering the sign of x, be turned into another with a real positive root, when the geometrical method again becomes applicable. It will also, as we shall see, enable us to represent both roots when both are real and positive, and therefore to represent both roots when both are real but either positive or negative.[126]

This blatantly algebraic interpretation, then, sees the significance of VI.27-29, the very culmination of "application of areas," as lying in the Greek solution to the general quadratic equation. Later, we will spell out the precise grounds upon which Heath's interpretation rests, and offer a critique of his position; here we only wish to register our dissent with it. But, although we emphatically disagree with Heath's rationale for thinking VI.27-29 so significant, we are, nevertheless, in complete agreement (at least in spirit)[127] with his subsequent remarks:

> The method of these propositions was constantly used by the Greek geometers in the solution of problems, and they constitute the foundation of Book X. of the Elements and of Apollonius' treatment of the conic sections.... [Heath then quotes Simson's views on this matter]... "These two problems [VI.28 and 29], to the first of which the 27th Prop. is necessary, are the most general and useful of all in the Elements, and are most frequently made use of by the ancient geometers in the solution of other problems ...[128]

Let us now consider the enunciation and the main ingredients in the proof of Proposition VI.29:

> To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one.[129]

For the proof, we are given straight line AB, rectilineal figure C, and parallelogram D (Fig. III.4). First we bisect AB at E (by I.10). Next we utilize VI.18[130] to construct a parallelogram BF, similar and similarly situated to D, on EB. Now for the crucial step wherein VI.25 comes into play: we construct parallelogram GH equal to the sum of BF and C but similar to D. By producing FL to M and FE to N, so that FM = KH and FN = KG (which clearly can be done), and by completing MN, we obtain the figure pictured above where MN is both equal and similar to GH. Now VI.21[131] implies that MN is similar to EL, hence, by VI.26,[132] MN and EL are about the same diameter, FO. AN equals NB, by I.36,[133] while NB equals BM by I.43.[134] Since from the above it follows that gnomon XWV is equal to C, it is necessary that AO be also equal to C. Finally, by VI.24,[135] BO is similar to EL, and hence similar to D. Therefore, we have applied the desired parallelogram AO, to AB. For AO is equal to C and the "excess," parallelogram BO, is similar to D, q.e.f. We will return to this important result momentarily, but first we must pursue the analysis of our second chain of ideas.

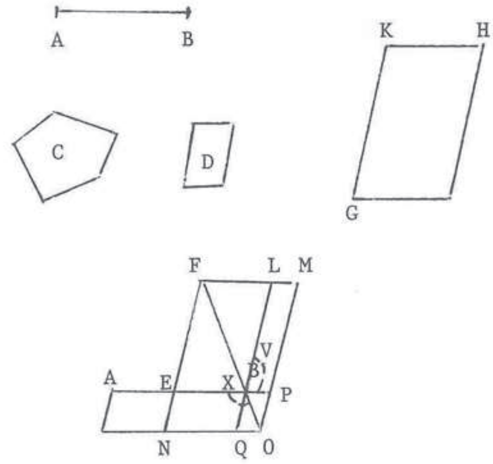



Fig. III.4

5. The second chain consists of the following theorems:
2) I.45 - I.47 - II.5 and 6 - II.11 and 14 - VI.27-29. As we have already given a detailed account of I.45, we turn to I.47, the "Pythagorean theorem":

> In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.[136]

The idea motivating Euclid's proof, which is carefully fashioned so as to avoid the use of proportion theory, can be seen in the "windmill" diagram shown at right (III.5). The argument shows that BL equals AF and LC equals AK, hence BL plus LC (which equals BE) also equals AF plus AK as required. To show that BL equals AF, observe that BL = 2(ABD) and AF = 2(FBC) by I.41,[137] while the triangles ABD and and FBC are equal because they are congruent by I.4.[138] It follows that BL = AF. Exactly the same reasoning shows that LC = 2(ACE) and AK = 2(BCK), whereas triangles ACE and BCK are equal, hence LC = AK. This completes the proof.

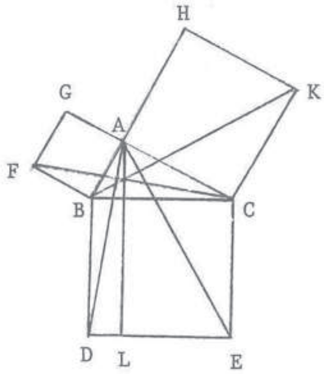


Fig. III.5

Proclus seems to credit Euclid himself with this very pretty proof, and there is good reason to believe that he is right.[139] For, by using results from the proportion theory of Book VI, one can obtain a very straightforward proof of this famous theorem.[140] It is therefore not unreasonable to think that the author of the Elements, who clearly required I.47 in several of the arguments appearing in Book II, i.e., prior to the introduction of proportion theory, was motivated to discover this ingenious argument by the organizational dictates he sought to follow in writing his great masterpiece. For, by delaying the introduction of proportion theory until Book V, he made an immense contribution to the creation of a streamlined presentation of the fundamental results of Greek geometry, a presentation that has served as the supreme model of elegant reasoning for over two thousand years.

Turning now to Propositions II.5 and II.6, we are once again presented with a pair of related results, the first analogous to VI.28, the second to VI.29. Proposition II.5 reads:

> If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.[141]

Even more important, for our purposes, is Proposition II.6, as it ties in with the results leading up to VI.29. Proposition II.6 reads:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.[142]

Both of these propositions strike the unwary reader as terrifically confusing and have the outward appearance of difficult theorems. But, in reality, they are very simple results that only look complicated because the Greeks were forced to render their mathematics in ordinary language, enunciating all the operations leading to the final diagrammatic construction in complete sentences, rather than merely starting with the finished product and arguing from it.[143]

In II.5, the final figure looks like this (Fig. III.6): AB has been cut in two equal pieces: AC and CB (∴ AD ≠ DB). II.5 asserts that DK plus EH is equal to BE (the last two rectangles are squares). But this is obvious (once we have the diagram!). For CH equals HF by I.43,[144] so if DM is added to each, we see that CM equals DF. Hence AL (which is equal to CM) equals DF, and by adding CH to both we have DK equals gnomon NOP.

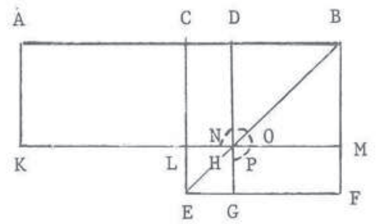



Fig. III.6

Finally, we add in the square EH and conclude that DK plus EH equals BE as desired. The key observation in the proof is a simple one, namely, that the rectangle AH equals the gnomon NOP because their intersection, CH, subtracted from both leaves equal rectangles (AL = DF).

Exactly the same considerations are involved in the proof of II.6 where the diagram (Fig. III.7) looks like this:

Again, AB is the given line, bisected at C and extended to D, and the theorem asserts that rectangle AM plus square EH equals square ED. And, once again, the key observation in the proof involves a gnomon. Here rectangle AM equals gnomon NOP, since their intersection, subtracted from both, leaves AL and HF which are equal. Thus by adding EH to both AM and the gnomon we obtain the desired result: AM plus EH equals ED.

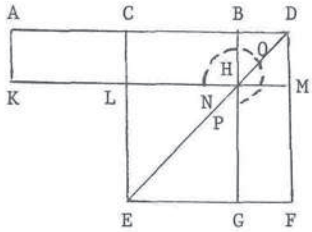



Fig. III.7

Much more could be said here concerning II.5 and 6, as both have been heavily exploited by the practitioners of "geometric algebra."[145] But we must pass on, and turn now to the first alleged instance in the Elements of a solution to a quadratic equation, Proposition II.11:

> To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.[146]

For the proof, apply I.46[147] to describe square ABDC (Fig. III.8) on the given line AB. Bisect AC at E, using I.10, and let BE be joined. Now produce CA to F so that EF equals EB and describe the square AFQH on AF. Then, the claim is that AB has been cut at H so that the rectangle contained by AB, BH is equal to the square on AH.

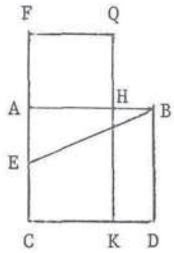



Fig. III.8

Confirming this claim requires both I.47 and II.6. Since AE equals EC, II.6 applies, hence the rectangle on CF, FA together with the square on AE is equal to the square on EF. But I.47 says that the square on BE is equal to the sum of the squares on AB and AE, and since BE equals EF, it follows that (1) the rectangle on CF, FA together with the square on

AE is equal to the sum of the squares on AB and AE. Thus by subtracting the square on AE from both equal expressions in (1) above, the rectangle on CF, FA (i.e., CQ) is equal to the square on AB. If now rectangle AK (which is common) is subtracted from both of these, we have that the square on AH is equal to the rectangle HD = rectangle on AB, BH as desired.

Proposition II.14 is another result that is believed to have been motivated by the desire to solve a quadratic equation. We are told by Heath and others that this result gives the Greek method for extracting "square roots." The proposition reads:

To construct a square equal to a given rectilineal area.[148]

The proof begins by applying I.45 to obtain a rectangle BD equal to the given figure A (Fig. III.9). If the rectangle turns out to be a square, we are done. If not, produce the longer side, BE, to F so that EF equals ED. Bisect BF at G (I.10) and form the semicircle BHF on BF as diameter. Extend DE to H where it intersects the semicircle. Since BG equals GF, II.5 applies and asserts that the rectangle on BE, EF, together with the square on GE, is equal to the square on GF. Now GF equals GH. Thus by utilizing I.47 to assert that the square on GE plus the square on HE equals the square on GF, while remembering that the square on FG is also equal to the rectangle on BE, EF plus the square on GE, and subtracting the square on GE from both sides, it follows that the rectangle on BE, EF (which equals A) is also equal to the square on HE (or, in symbols, [Rect. (BE,EF) + Sq.(GE) = Sq.(GF) = Sq.(GH) = Sq.(HE) + Sq.(GE)]⇒ Rect. (BE,EF) = Sq.(HE). Thus a square can be constructed, namely that on HE, equal to the given rectilinear figure A.

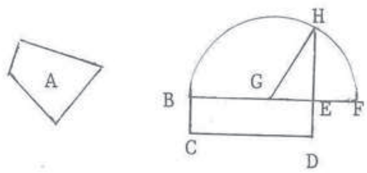



Fig. III.9

Heath sees II.14 as the culminating result in the "geometric arithmetic":

> As II.12, 13 [for the modern mathematician the geometrical analogues of the law of cosines] are supplementary to I.47, so II.14 completes the theory of <u>transformation of areas</u> so far as it can be carried without the use of proportions. As

> we have seen, the propositions I.42, 44, 45 enable us to construct a parallelogram having a given side and angle, and equal to any given rectilineal figure [this is I.45A]. . . . Further, I.47 enables us to make a square equal to the sum of any number of squares or to the difference between any two squares. The problem still remaining unsolved is to transform any rectangle (as representing an area equal to that of any rectilinear figure) into a square of equal area. The solution of this problem, given in II.14, is of course [!] the equivalent of the extraction of the square root, or of the solution of the pure quadratic equation
>
> $$x^2 = ab.$$ [149]

Implicit in this interpretation of II.14 as the extraction of a square root or as the solution of $x^2 = ab$, is the view, which for us is clearly untenable, that the Greek "operation" of rectangle formation was seen by the Greeks as multiplication of general magnitudes. Once this latter assumption is rejected, it becomes obvious that II.14 can no longer be seen as "equivalent" to the solution of a quadratic equation. Rather, the motivation for II.14 seems to stem primarily from proportion theory, as, basically, the same construction is applied to prove VI.13: "To two given straight lines to find a mean proportional."[150] Indeed, Aristotle remarks in this connection that squaring should be better defined as the finding of the mean proportional rather than the making of a square equal to a given rectangle, because the former gives the _cause_ of the result, whereas the latter gives the _conclusion_ only.[151]

We are now ready to forge the final link in our second chain of propositions by showing the connection between II.6, II.11 and VI.29. This last link can be seen best by returning to the proof of the last of these propositions, VI.29. The first chain of results, I.42 - I.45 - VI.25 - VI.28 and 29, that we examined led straightaway to one of the two key ideas in the proof of VI.29, namely the application of VI.25. The second chain, I.45 - I.47 - II.5 and 6 - II.11 and 14 - VI.27-29, leads to the second key idea behind VI.29, but the connection this time is much more subtle. Once VI.25 had been applied in the proof of VI.29 (_cf._ Fig. III.4), it was a routine matter to arrive at the following figure (III.10):

The remainder of the proof involved the assertion that the gnomon XWV is equal to AO. But this is exactly the key ingredient in the proof of II.6, when it is generalized to a parallelogram.[152] What is more, there is absolutely no difficulty in extending II.6 to the more general result (II.6A) involving parallelograms, as all the necessary ingredients for doing so are already available in Book I! For, since AO intersects gnomon XWV in EO, it suffices to show that AN equals BM. But AN equals NB by I.36[153] and NB equals BM by I.43,[154] so AN equals BM, and therefore AO equals XWV as desired. To complete the proof of II.6A, simply add EL to both figures obtaining AO plus EL equals MN.

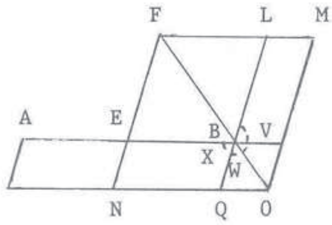



Fig. III.10

Thus, II.6, or more precisely our II.6A, is the second key idea in the proof of VI.29. We have seen already that II.6 is a key ingredient in the proof of II.11, but what is the connection between II.11 and VI.29? Simply this, II.11 corresponds to the special case of VI.29 wherein the "excess" is a square.[155] If we return for a moment to Fig. III.8, we shall notice that if AC rather than AB is taken as the given line, then II.11 can be interpreted as the application to AC of the parallelogram CQ equal to the rectilinear figure AD and exceeding by the Figure AQ similar to a given square figure. Since II.14 can be used to find a square (viz. AD) equal to a given rectilineal figure, it will be observed that, on this interpretation II.11 is precisely the special case of VI.29 wherein the "excess" is required to be similar to a given square!

One final observation: almost everything that has been said here concerning the relationship between II.6 and VI.29 carries over verbatim to the case of II.5 and its cognate result, VI.28. Moreover, VI.28 itself is entirely analogous in format with VI.29, except for the fact that it is an application that "falls short" rather than "exceeding":

> To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a

> parallelogrammic figure similar to a given one: thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.[156]

The qualifying clause at the end of the proposition refers to VI.27 which is a διορισμός.[157] It reads:

> Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.[158]

VI.27 says, then, that the largest area that can be obtained via an application that "falls short" occurs when the applied parallelogram utilizes, for its base, half the given line. Clearly if VI.28 is to have a solution, the given rectilineal figure in VI.28, to which the applied parallelogram is to be made equal, must not exceed the maximum area such a parallelogram can attain, and this maximum area is known from VI.27.

This completes our survey of the key ideas involving the technique of "application of areas" as they appear in Euclid's Elements. A more thorough study of this subject would not overlook the important applications of this technique made by the Greeks. In Book X, for example, it is used in the proofs of four important theorems: implicitly in X.17 and X.18, explicitly in X.33 and X.34, both of which depend on VI.28.[159] The most prominent example is, of course, Apollonius's use of "application of areas" in Propositions 11-13 of Book I of his Conics to derive the symptoma of the conic sections.[160] As is well known, these curves were named by Apollonius "hyperbola," "parabola," and "ellipse" as they correspond to the three cases of "exceeding," "application," and "falling-short" in the terminology of "application of areas."

6. Having discussed the key ideas that appear in the Elements pertaining to "application of areas," it is time to deal with their alleged use as solutions to quadratic equations. In this regard, we shall now consider the algebraic interpretation of Proposition II.11:

Let $AB = a$ in Fig. III.11, then we wish to find $x$ so that $0 < x < a$ and $a(a-x) = x^2$ or $x^2 + ax = a^2$. Now in II.11, AC is bisected and the right triangle

ABE is formed. If we let BE = c, then, by I.47, $c^2 = a^2 + (a/2)^2$. Since BE = FE, we obtain the desired x by forming FA = FE minus AE. Hence $x = c - (a/2)$. To check that this x works, we use II.6 which says that $x(x+a) + (a/2)^2 = c^2$; since $c^2 = a^2 + (a/2)^2$ we obtain $x(x+a) + (a/2)^2 = a^2 + (a/2)^2$, whence $x^2 + ax = a^2$ as desired.

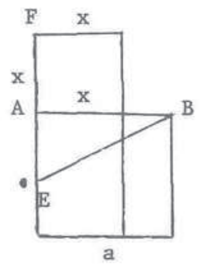



Fig. III.11

Now Heath, following Simson, shows that the algebraic formulation of this proposition, as well as the proof, generalize line for line to give a solution to the equation $x^2 + ax = b^2$.[161] For let ——a—— and ————b———— be given, Fig. III.12, and let $c^2 = b^2 + (a/2)^2$ using I.47; then $x = c - (a/2)$ is the desired solution. For by II.6, $x(x+a) + (a/2)^2 = c^2 = b^2 + (a/2)^2$, hence $x(x+a) = x^2 + ax = b^2$.

But what Heath is careful <u>not</u> to do is to take his own views on the "geometrick arithmetic" seriously in this connection, for had he done so, he would have soon found that the so-called Greek algebra, i.e., "geometric algebra," is nothing like our own (it's not very Greek either!). As we have previously seen, Heath's conception of the "geometric arithmetic" is largely motivated by the belief that, in much of Greek mathematics, geometric considerations are subordinate to questions of pure magnitude. This is the fundamental fallacy that constantly lurks behind the double phantasm of "geometric arithmetic" and "geometric algebra." It is true that, for the most part, this viewpoint remains rather submerged throughout Heath's commentary,[162] but occasionally it surfaces fairly clearly, as in his remarks pertaining to the "great importance" of I.44 and I.45 as a means for representing a given rectilineal figure as a rectangle "with one side of any given length, e.g., a <u>unit</u> length."[163] As we pointed out in section I, this "great importance" seems, however, to have been lost on the Greeks, because there is not a shred of hard evidence indicating that they ever employed these propositions for this (Heath's)

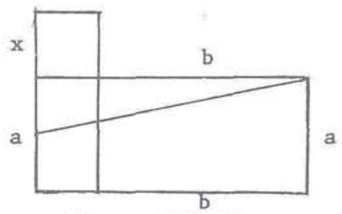



Figure III.12

purpose.[164] Simply put, the Greek geometers had no conception of magnitude as generalized number, and, what is more, their interest in magnitude was (except in Book V) always bound to a specific geometric context.

Let us, however, just for the sake of argument, take Heath's position at face value in order to see the kinds of difficulties it leads to. It is our contention that if one is motivated to solve quadratic equations involving pure magnitudes, it is a simple enough matter to apply the technique of II.11 to solve, for example, the equation $x^2 + ab = b$! Of course, this equation makes no sense geometrically, since the left side is two-dimensional, whereas the right side is only one-dimensional. If, however, we view this, à la Heath, as a statement about pure, dimensionless magnitudes, then it not only makes sense, but it can even be handled with complete ease, utilizing "Greek" tools. For this same equation can be translated into an equivalent one that does make sense geometrically,

$$x^2 + ax = b.1,$$

where $b.1$ represents the rectangle with one side of length $b$ and the other of unit length! Now simply apply II.14 to obtain a square, call it $c^2$, equal to the rectangle $b.1$ and we have:

$$x^2 + ax = c^2,$$

which is precisely the equation we solved a moment ago by mimicking the proof of II.11.

Of course, this is pure fantasy, and neither Heath nor anyone else would blunder so badly as to mistake this for a Greek solution to a quadratic equation. The point, however, is (and it is a point worth emphasis) that such a solution is perfectly plausible once we take the assumptions of "geometric algebra" seriously. This "reconstruction" is perfectly consistent with the dictates of "geometric algebra," which entitle us, indeed require us, to couple freedom of expression (form) with a virtually total lack of concern for the ontological commitments inherent in Greek mathematics. If one can, in effect, ignore the dimension of the magnitudes involved when "multiplying," then why not do it here?

The tendency to formulate this kind of reconstructive argument has, indeed, run rampant in the literature of the history of ancient mathematics, and although such arguments have gained an honorable place in the discipline, they are gounded on assumptions that are radically different from the prevailing thought of the culture under consideration and ought, therefore, to be clearly recognized as having no historical value. But, as we are presently engaged only in the process of exploding an ahistorical myth, our only obligation for the moment is to play the game according to the prevailing rules; so let us pursue this fantasy one step further. This time, and again we are using the assumptions of "geometric arithmetic" alone, we will solve the general quadratic,

$$ax^2 + bx = c,$$

where the magnitudes are, of course, all positive (lines).

The agenda is the same as before; first we must convert the equation to an "equivalent" one that makes sense geometrically. Write $bx = (b.1)\ x$ and $c = c.1^2$, where $b.1$ is the rectangle with the sides of length $b$ and $1$ as before, while $1^2$ is the unit wquare, then we obtain:

$$ax^2 + (b.1)\ x = c.1^2$$

which makes sense geometrically. Now use I.45A to write $b.1$ as $a.p$:

$$ax^2 + (a.p)\ x = c.1^2$$

Using now the three-dimensional analogue of II.1[165] we obtain:

$$a\ [x^2 + px] = c.1^2$$

Since equals divided by equals are (presumably) equal:

$$\frac{a\ [x^2 + px]}{a} = \frac{c.1^2}{a} = \frac{c.1}{a}\ .\ 1$$

Again, using I.45A, $c' = \frac{c.1}{a}$ and we have:

$$x^2 + px = c'.1$$

Finally apply II.14 to transform $c'.1$ to $d^2$ and now:

$$x^2 + px = d^2$$

which is solved again using the proof in II.11. We have thus "solved" the general quadratic by using nothing more than Book II-style techniques, i.e., bypassing altogether Greek proportion theory. Now <u>that</u> is geometric algebra!

7. Let us now get our feet at least halfway back on the ground again by returning to the algebraic interpretation of some propositions that actually occur in the <u>Elements</u>. We have seen that II.11 is a special case of VI.29 and that it also "corresponds" to the equation $x^2 + ax = a^2$. We shall now indicate what is involved in interpreting Propositions VI.28 and VI.29 algebraically, i.e., seeing them as the

Euclidian versions of the solutions to the general quadratic equations $ax \mp \frac{b}{c} x^2 = C$ respectively. Recall that in VI.28 and 29 we are given (Fig. III.13) a line AB, a rectilinear figure C, and a parallelogram D, and we are asked to construct (Fig. III.14) a parallelogram equal to C on the base AE (where A, B, and E are co-linear) so that either the "defect" (VI.28), or the "excess" (VI.29), i.e., the parallelogram in the same parallels as the construced parallelogram, but with base BE, is similar to the given parallelogram D.

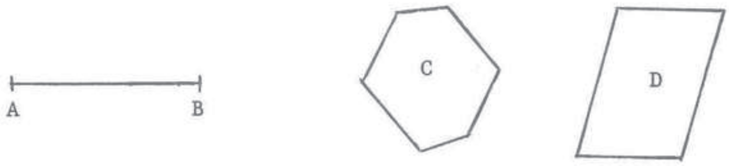



Fig. III.13

The interpretation of these propositions as algebraic equations requires, first of all, that we make the restriction that the parallelogram D be a rectangle.[166] This restriction is motivated by the desire to express the "defect" or "excess" in VI.28 and 29 as a "product," using the fallacious rectangular representation which, as we have already argued, never meant "multiplication" to the ancient Greeks. If, now, the ratio of the sides in the given rectangle D is b:c, then this must also be the ratio of the sides of the "defect" ("excess"). By letting one side be x (cf. Fig. III.15), and using VI.12,[167] we can solve for the fourth proportional to obtain the other side, say y. Thus

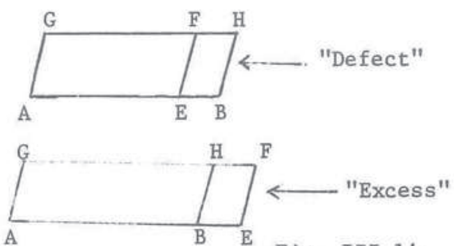



Fig. III.14

$$b:c = y:x$$

and, by VI.16, Rect. (b,x) = Rect. (c,y) which in the "geometric arithmetic" reads

$$bx = cy.$$

Dividing both sides by c (use I,45A) we obtain

$$y = \frac{bx}{c} .$$

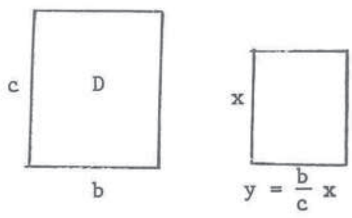



Fig. III.15

Now, to measure the "defect" ("excess") we multiply:

$$y \cdot x = \frac{bx}{c} \cdot x.$$

But $$\frac{bx}{c} = \frac{b}{c} \cdot x,$$ [168] hence

$$y \cdot x = \frac{b}{c} x^2.$$ [169]

Plugging all this into the framework of VI.28 and 29, with the given line AB = a we obtain (cf. Fig. III.16):

VI.28: Parallelogram AEFG - Rectilinear Figure C

$$ax - \frac{b}{c} x^2 = C$$

VI.29: $$ax + \frac{b}{c} x^2 = C$$

which are the quadratic equations II.11 corresponding to the case of VI.29 where the "excess" is a square.

If we review now this procedure carefully, it would seem that it is not without some difficulty that the statements of VI.28 and 29 can be interpreted as equivalent to quadratic equations. As noted above, one must first restrict these theorems to the case where the given parallelogram D is a rectangle. Moreover, one must be careful about expressing the appropriate equation in its "proper" form. Obviously, the usual form for the general quadratic, $ax^2 + bx + c = 0$, will not do, as a, b, c, and x must all be nonnegative. Neither will it do to express it in the form $ax^2 + bx = c$, as $ax^2$ is three-dimensional, bx two-dimensional, and c only one-dimensional. The above interpretation manages to "overcome" this difficulty by writing the general quadratic in the form $ax \mp \frac{b}{c} x^2 = C$. Here the three coefficients in the equation are represented by three mathematically distinct entities: <u>a</u> is a magnitude associated with an arbitrary line segment, <u>C</u> a magnitude represented by an arbitrary rectilineal figure, and $\frac{b}{c}$ is the ratio of two given lines (namely the sides of the rectangle D "given" in VI.29).

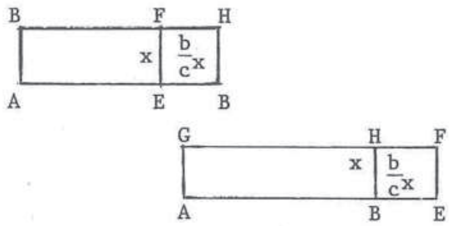


Fig. III.16

The term $\frac{b}{c} x^2$ is the most problematic feature in this arrangement. It requires that we view ratio as a sort of pure, dimensionless magnitude, and seems to imply that ratios are capable of being multiplied with magnitudes of definite dimension. The immediate difficulty this presents stems from the fact that multiplication of a ratio by a magnitude is never defined and never performed anywhere in the

Elements (or in all of Greek geometry, so far as we know!). The most plausible attempt to make sense of $\frac{b}{c}x^2$ in Greek terms is to follow De Morgan and "[t]reat ratio . . . as an engine of operation. Let that [ratio] of A to B suggest the power of altering any magnitude in that ratio."[170] This seems to be in tune with the spirit of the Greek view, which would utilize the fourth proportional to find the geometrical entity corresponding to $\frac{b}{c}x^2$ as follows: We are given $b:c$ and $x^2$, hence we must find a two-dimensional magnitude $Y$ such that $b:c = Y:x^2$. This would seem to be a simple enough matter, for, by VI.12,[171] we can find a line $p$ so that $b:c = p:x$. Thus, by VI.1,[172] $p:x$ = Rect. $(p,x)$:Sq. on $x$, so $\therefore$ $b:c$ = Rect. $(p,x)$:Sq. on $x$, and we have found our $Y$(= Rect. $(p,x)$). This easy argument, twisted à la grecque, would appear to answer any doubts one might have about the Greek interpretation of $\frac{b}{c}x^2$, except for one embarrassing detail (which, by now, should sound familiar)--it never appears anywhere in the Elements.[173] Thus, the key proposition that would enable us to see the "real meaning" of Propositions VI.28 and 29 must itself be reconstructed, as the only format in Euclid for finding the fourth proportional is that given in VI.12, where its use is restricted to lines. Certainly, if this more general version for finding the fourth proportional had played such an important role in what Simson called the "most general and useful [problems] of all in the Elements," i.e. VI.27-29, it is, indeed, difficult to understand how it could have been completely omitted.

Of course, none of this causes any difficulty for the practitioners of "geometrical algebra." Their approach is simply to ignore the constraints of the Greek approach to ratio and proportion, and proceed along their merry, modern way. One moment we have a pure, dimensionless quantity called ratio, the next moment a concrete representation by line segments which can be manipulated via the method of "application of areas."[174] To make sense of $\frac{b}{c} \cdot x^2$ using this approach, we have a number of options open to us! The argument already given is one; another is to use three dimensions. The fact that $\frac{b}{c} \cdot x^2 = \frac{bx^2}{c}$ suggests that the way to proceed would be to transform the rectangular prism $bx^2$ into an equal prism, but with one side of length $c$. But, for the geometrical algebraist, this is utterly trivial. All one has to do is apply I.45A to obtain the rectangle $cd = x^2$. Thus $bx^2 = bcd$, and $\frac{bx^2}{c} = \frac{bcd}{c} = bd$! Of course, again, nothing like this appears anywhere in the Elements, wherein three-dimensional techniques involving transformation of volumes seem to be altogether absent.[175] Still it is very hard to imagine why, if the Greeks in fact possessed a "geometric arithmetic," they should have restricted its use to the plane. If a line times itself was a wquare, surely one could "multiply" again and obtain a cube![176]

There is another option that "geometric arithmetic" makes available in dealing with $\frac{b}{c} \cdot x^2$, namely, to invert the order of operations: first divide $x^2$ by c, and then multiply by b. This has the added advantage that the operations are performed, à la Heath, without ever having to leave the plane. Thus we see that, using "geometric arithmetic," one has all the flexibility in the world, and a problem like finding $\frac{b}{c} \cdot x^2$ when given b:c and $x^2$ is, in effect, just about as natural as manipulating the expression algebraically! And indeed this is the reason for the irresistible appeal carried by "geometric arithmetic" with "geometrical algebraists."

But let us look back a minute. We have seen that by restricting VI.29 to the special case where the given parallelogram is a rectangle, by representing each of the coefficients in the equations $ax \mp \frac{b}{c} x^2 = C$ by a different geometric entity, and by employing dubious reasoning to interpret what is meant by $\frac{b}{c} \cdot x^2$, one can indeed reconcile these equations with the statement of Propositions VI.28 and 29. Notice that nothing has yet been said concerning the solution of these equations using VI.28 and 29, although we are assured by Ivor Bulmer-Thomas that ". . . the geometrical method is precisely equivalent to the algebraical method of completing the square. . . ."[177] We will consider the merits of this viewpoint momentarily, but here we would simply like to say that judging from the above, it is not without considerable effort that one can reconcile even the enunciations of VI.28, 29 with the equations $ax \mp \frac{b}{c} x^2 = C$, let alone the proofs of these statements.

8. Let us now take a look at the solutions of the above equations using VI.28, 29 as a guide. According to Heath, we should find an ". . . exact correspondence between Euclid's geometrical and the ordinary algebraical method of solving the equation[s] . . . ."[178] But if we naively set about to solve $ax + \frac{b}{c} x^2 = C$ algebraically, we will see that the correspondence is far from "exact." For convenience let $C = d^2$, by using II.14, then, algebraically:

$$ax + \frac{b}{c} x^2 = d^2$$

$$x^2 + \frac{ca}{b} x = \frac{cd^2}{b}$$

and completing the square, $x^2 + \frac{ca}{b} x + \frac{c^2a^2}{4b^2} = \frac{cd^2}{b} + \frac{c^2a^2}{4b^2}$

$$\left(x + \frac{ca}{2b}\right)^2 = \frac{cbd^2}{b^2} + \frac{c^2a^2}{4b^2}$$

$$x + \frac{ca}{2b} = \frac{\pm\sqrt{4cbd^2 + c^2a^2}}{2b}$$

$$x = \frac{-ca \pm \sqrt{4cbd^2 + c^2a^2}}{2b}$$

If, now, we look again at the proof of VI.29 (making all the necessary algebraic substitutions, etc.) it soon becomes apparent that there must be some other "exact" correspondence, for clearly this one does not work! Of course, anyone who knows the first thing about Greek mathematics would have recognized at once that this naive, but straightforward, algebraic approach was doomed to failure right from the start. For there is nothing in Greek mathematics that bears even the slightest resemblance (no matter how one looks at it) to the operation of multiplying both sides of a geometric equality by a general ratio.[179] So what, pray tell, is the correspondence that Heath, Thomas, and others have in mind? To answer this, we must first translate the proof of VI.29 into algebraic terms.

The first step in proving VI.29 was to construct (see Fig. III.17) parallelogram EL similar to D. Supposing now all the parallelograms to be rectangles, $EL = \frac{a}{2} \cdot \frac{c}{b} \cdot \frac{a}{2} = \frac{a^2c}{4b}$. Next we used VI.25 to construct parallelogram (now rectangle) MN equal to EL plus $C = (d^2)$ and similar to D. Thus, as can be seen from the diagram, we have $MN = \frac{a^2c}{4b} + d^2 = \left(\frac{b}{c}x + \frac{a}{2}\right)\left(x + \frac{ac}{2b}\right)$. Now to obtain x we must compute FN by using the fact that $(FN)\left(\frac{b}{c}FN\right) = \frac{a^2c}{4b} + d^2$. Thus $FN = \sqrt{\frac{c}{b}\left(\frac{a^2c}{4b} + d^2\right)}$ and $x = FN - FE = \sqrt{\frac{c}{b}\left(\frac{a^2c}{4b} + d^2\right)} - \frac{ac}{2b}$.[180]

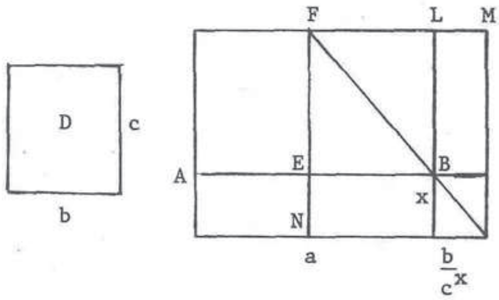



Fig. III. 17

The corresponding algebra is obtained by completing the square in the equation $ax + \frac{b}{c}x^2 = d^2$ without altering the coefficients:

$$\frac{b}{c}x^2 + ax + \frac{c}{b} \cdot \frac{a^2}{4} = d^2 + \frac{c}{b} \cdot \frac{a^2}{4}$$

$$\left(\sqrt{\frac{b}{c}}\, x + \sqrt{\frac{c}{b}} \cdot \frac{a}{2}\right)^2 = d^2 + \frac{a^2c}{4b}$$

whence,

$$\sqrt{\frac{b}{c}}\, x + \sqrt{\frac{c}{b}} \cdot \frac{a}{2} = \sqrt{d^2 + \frac{a^2c}{4b}}$$

$$\frac{b}{c}\, x = \sqrt{\frac{a^2c}{4b} + d^2} - \sqrt{\frac{a^2c}{4b}}$$

$$x = \sqrt{\frac{c}{b}\left(\frac{a^2c}{4b} + d^2\right)} - \frac{ac}{2b}$$

as above.

And now what is the "exact correspondence" between this solution and Euclid's? According to Heath, it is due to the fact that Euclid geometrically obtains the quantity $\frac{a^2c}{4b}$ (the parallelogram--actually the rectangle--EL) in order to "complete the square." This seems to make sense when we look at the diagram and take note of the fact that VI.25 is employed precisely in order to produce a gnomon equal to the given rectilineal figure C that "fits" around parallelogram EL, thus "completing" the _figure_. But does this correspondence go any further?

In algebra, the technique of completing the square has a definite _object_, namely to factor the equation in the form $(x + a)^2 = b$, whereupon taking square roots of both sides, a solution is obtained. Thus completing the square, i.e., transforming the equation $px^2 + qx = r$ into the form $(x + a)^2 = b$ has as its only _raison d'être_, the possibility of extracting square roots as the next step in the procedure. What we find in the proof of VI.29, needless to say, is nothing of the kind! There are no squares in the proof of VI.29 so, to begin with, it is a misnomer to speak in this connection of "completing the square." But beyond this, the whole idea behind completing the square is totally foreign to the method of proof found in Euclid. The only "factorizations" that one can transcribe from the set-up in VI.29 (once one is no longer dealing with general parallelograms) involve, as we saw, rectangles, e.g., $(\frac{bx}{c} + a/2)(x + \frac{ac}{2b}) = \frac{a^2c}{4b} + d^2$, not _squares_. If VI.29 involved "completing a square," one would expect to find II.14 (the so called "equivalent of the extraction of the square root")[181] employed in the argument, but, of course, nothing even mildly resembling II.14 is ever used. Perhaps the best way to make our point is to give an illustration of what "completing the square" really looks like geometrically. In doing so, another interesting question arises, namely, why is there nothing comparable to the following simple solution of a quadratic equation anywhere in Greek mathematics, if, as claimed, the Greeks solved equations geometrically?

To solve $x^2 + bx = C$, first apply II.14 to get $d^2 = C$. Next "complete the square" as follows:

$x^2 + bx = x^2+2(\frac{b}{2}x) = d^2$, hence, by II.4[182], adding $(\frac{b}{2})^2 = \frac{b^2}{4}$ to both sides produces a perfect square, i.e., $x^2 + 2(\frac{b}{2}x) + \frac{b^2}{4} = (x + \frac{b}{2})^2 = d^2 + \frac{b^2}{4}$. Geometrically this amounts to completing the diagram below (cf. Fig. III.18).

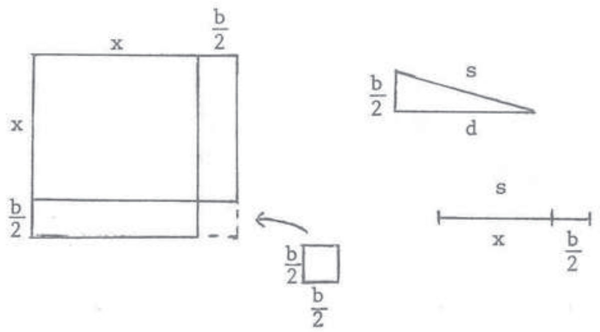


Fig. III.18

Using I.47, we can find $s$ such that $d^2 + (\frac{b}{2})^2 = s^2$ simply by constructing a right triangle with $d$ and $b/2$ as sides. It follows that $(x + \frac{b}{2})^2 = s^2$, hence $x + \frac{b}{2} = s$, and $x = s - b/2$.

As long as we are at it, we might as well make the additional observation that this technique (mixed with a pince of "geometric arithmetic") will give a solution to the general quadratic $ax + \frac{b}{c}x^2 = C$. The idea here amounts to transforming the given equation into another equation whose "squared" term has a coefficient of one, using what van der Waerden, et al., call a change of variable,[183] Thus $x^2 = yc$, by I.45A, hence $\frac{b}{c}x^2 = \frac{byc}{c} = by$. Now use II.14 to obtain $z^2 = by$, and I.45A again to obtain $dz = ax$. This gives us the equation $z^2 + dz = C$, which we solve as above. Having found $z$, we then work backwards noting that $\frac{b}{c}x^2 = by = z^2$, which is known. Thus $x^2 = \frac{c}{b}z^2$, hence $x$ is known, by II.14.

There is another, even simpler, solution to the general quadratic available to us, based on an idea of Heath's. Recall that in section II, #5 we discussed Heath's theory that the Greeks utilized ratios instead of magnitudes in order to solve algebraic equations, or, more generally, in order to circumvent the limitations imposed by the necessity of having to manipulate magnitudes in the context of two-, or at most three-dimensional space. This theory would seem to suggest that the Greeks recognized the "validity" of propositions like II.1-10 when applied to ratios, rather than just to line segments. Indeed, the transference would seem to be automatic, since, according to Heath, the Greeks viewed the operation of forming the rectangle on two given lines as being completely equivalent to the operation of compounding their "corresponding" ratios. But if this is true, the Greeks could also have solved the equation $ax^2 + bx = C$ by doing exactly what Heath claims is done (but, in fact, is not done) in VI.29, namely, by completing the square without altering the coefficients. Thus

$$ax^2 + bx = c$$

$$ax^2 + bx + \frac{b^2}{4a} = c + \frac{b^2}{4a}$$

$$(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = c + \frac{b^2}{4a} = (\sqrt{c})^2 + (\frac{b}{2\sqrt{a}})^2 = s^2$$

$$\sqrt{a}x + \frac{b}{2\sqrt{a}} = s$$

$$\sqrt{a}x = s - \frac{b}{2\sqrt{a}}$$

$$x = \frac{s - \frac{b}{2\sqrt{a}}}{\sqrt{a}} .$$

Notice that there is nothing problematic about interpreting any of these symbols because they are ratios--pure, dimensionless quantities that require no geometric representation. The justification for the steps in the argument depends upon nothing more than taking Heath's views seriously, and simply putting them into practice. Thus writing $(\sqrt{c}\ )^2 = c$ simply requires a ratio-theoretic version of II.14, finding $s^2$, a ratio-theoretic version of I.47, dividing by $\sqrt{a}$, a ratio-theoretic counterpart to I.45A, and so on.

Again, this is fantasy, not history, certainly not history of Greek mathematics, but it clearly illustrates the kind of dangers that present themselves when one employs the full power of an unbridled "geometric arithmetic" dressed à la grecque to solve problems in "geometric algebra." It also shows that it is not at all difficult, using "geometric arithmetic," to develop a geometric analogue to the algebraic technique of completing the square. Quite clearly this is not the method of Proposition

VI.29!

There is still a good deal more that must be said concerning the algebraic interpretation of the proof of VI.29. If we reread the statement of VI.29, it will be observed that (unlike II.11, for example) nothing is said that would lead one to believe that the proposition is in any way concerned with finding something unknown, i.e., algebraically, obtaining an explicit solution for the unknown line segment $x$. Of course, finding $x$ is mathematically equivalent to constructing the desired parallelogram referred to in the enunciation of VI.29, but, as a matter of fact, if we reexamine the proof of VI.29, it will readily be seen that there is no reference to the line $x$ there either. Finally, if we turn to Heath's reconstruction, we begin to see how misleading his claim (that there is an "exact correspondence" between Euclid and the technique employed using algebra) really is. The "correspondence" consists, in fact, of one step in the algebraic solution to the equation! For, after constructing parallelogram $EL = \left(\frac{a^2c}{4b}\right)$, see fig. III.17 above, which is added to both sides of the equation $\frac{b}{c}x^2 + ax = d^2$, we obtain $\frac{b}{c}x^2 + ax + \frac{a^2c}{4b} = d^2 + \frac{a^2c}{4b}$. The next step <u>algebraically</u> is $\left(\sqrt{\frac{b}{c}}\,x + \sqrt{\frac{c}{b}} \cdot \frac{a}{2}\right)^2 = d^2 + \frac{a^2c}{4b}$, but <u>geometrically</u> this has absolutely nothing to do with the proof in Euclid!

This same remark holds true for the solution that finds $x$ by observing that $(FN)\left(\frac{b}{c}FN\right) = \frac{a^2c}{4b} + d^2$, hence $x = FN - FE = \sqrt{\frac{c}{b}\left(\frac{a^2c}{4b} + d^2\right)} - \frac{ac}{2b}$. One will not find it in Euclid! The fact is, and it is a significant fact that bears repeating, that neither the statement nor the proof of VI.29 have anything to do with finding a certain $x$ explicitly, rather they are concerned with the construction of a certain parallelogram. Moreover, the means employed for performing that construction (primarily VI.25) are very powerful and general, and, of course, they have to be in order to handle a proposition in which one of the givens is an arbitrary rectilineal figure. If, in fact, there were a good correspondence between VI.29 and the solution to the general quadratic equation, one would expect to find a clear-cut geometric procedure for constructing the unknown $x$, i.e., one would expect to find something geometrically comparable to the <u>quadratic formula</u>. What we find instead, is that not only is there no such procedure given in the proof, but the proposition itself appears to be only marginally concerned with producing that $x$ at all!

What should be, by now, manifestly clear is that the interpretation of Propositions VI.27-29 as solutions to algebraic equations has been made at considerable cost to their geometric content. But, before closing this discussion, we should observe one final feature in VI.29 that makes it highly unsuitable for "solving an equation," namely its use of the given rectilineal figure C. As the reader is by now well aware, the Greeks had Proposition II.14 at their disposal, so that, had they wanted to, they could have replaced the given rectilineal figure C by a square, say $d^2$. Now if this is done, it is a relatively easy matter to obtain an explicit construction for $x$ by computing FN - FE as above. Not that anything of the kind is done in Euclid; we are simply pointing out that it could have been done. The procedure for constructing $x$ involves nothing more than some straightforward applications of "geometric arithmetic," which we leave for the reader's recreation.

As was mentioned earlier, VI.29 makes no use of Proposition II.14; instead it uses the very general result of VI.25 in order to incorporate the given rectilineal figure C into a larger parallelogram. This makes the problem of finding an explicit construction for $x$ a most unwieldy chore, for it, first of all, requires that we go back to the proof of VI.25, which is badly botched in Euclid, and work it out correctly.[184] Since the proof of VI.25 ultimately depends on the triangulation of the given figure C, (involving the repetition of a procedure $n$ times depending on the number of sides of the rectilineal fig. C), it is apparent that using this approach to explicitly construct $x$ would be horrendously complicated.

The point is, why start with a rectilineal figure at all? If one is interested in solving a quadratic equation, then one is interested in magnitude (μέγεθος), not shape (μορφή), and since we have a theorem (II.14) that tells us that every rectilinear figure has the magnitude of a square figure, why not simply take $d^2(=C)$ as given and be done with all the headaches. The answer to this is, of course, very simple: Greek mathematicians were not interested in magnitude divorced from geometry, and, with the qualified exception of Book V, there is no theory of "pure magnitude" anywhere in the extant corpus of Greek mathematics.

The statement and proof of VI.29 are typical of what we find in Greek geometry, insofar as they illustrate the manner in which a proposition is enunciated and demonstrated via a chain of reasonings that relies on previously established general principles, i.e., elements (στοιχεῖα).[185] In this all important respect, the Greek method differs completely from the technique employed in solving a quadratic equation. For whereas symbolic algebra uses a sequence of explicit operations and symbolic manipulations to obtain an explicit result, i.e., a formula e.g., $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, the method typified by VI.29 produces only implicit constructions based on stoicheia. One can, of course, trace back through the maze of arguments and exhibit an explicit construction in each specific case, but doing so does not reveal the actual modus operandi of Greek geometry.

IV

1. In the course of our analysis of the situation regarding the alleged solution by the Greeks of quadratic equations via Euclid II.11, II.14, VI.28, and VI.29, we had occasion to consider a number of illusory, fanciful solutions in order to illustrate the dangers inherent in the process of rendering carelessly Greek geometric procedures into algebraic language. The specific culprit at the root of this confusion is "geometric arithmetic," which imputes to the Greek mode of procedure regarding ratios, magnitudes, and their respective arithmetic relations, a degree of flexibility, generality, and abstractness that it apparently never had. In this section we will pursue our analysis of this interpretive line still further by showing that if Pandora's box containing "geometric arithmetic" is opened all the way (and why shouldn't it?), the phantoms that emerge soon make a shambles out of any sane attempt to understand Greek mathematics in its own terms.

The first item on our agenda involves the possibility of extending the results of two-dimensional "geometric algebra" to the third dimension. As we remarked earlier, there is scarcely any evidence that Greek geometers ever extended the results of Book II or the "application of areas" to three dimensions (or the "application of volumes"). However, lest the reader surmise that we are about to engage on an imaginary and unwarranted leap into the wild blue yonder, we must hasten to point out that, according to B. L. van der Waerden, not only did the Greeks extend "geometric algebra" to three-space, but this was the very focus of the ancient Greek researches into solid geometry. Let us begin by examining the argument in detail. As evidence for the above claim, van der Waerden makes an oblique reference to an obscure passage from the Epinomis (a sequel to Plato's Laws), which, he assures us, is "counted as one of Plato's works but not published until after his death. . . ."[186] The passage in its full context reads as follows:

> Hence there will be a need for several sciences. The first and most important of them is likewise that which treats of pure numbers--not numbers concreted in bodies, but the

> whole generation of the series of odd and even, and the effects which it contributes to the nature of things. When all this has been mastered, next in order comes what is called by the very ludicrous name _mensuration_ (γεωμετρία), but is really a manifest assimilation to one another of numbers which are naturally dissimilar, effected by reference to areas. Now to a man who can comprehend this, it will be plain that this is no mere feat of human skill, but a miracle of God's contrivance. Next, numbers raised to the third power and thus presenting an analogy with three-dimensional things. Here again he assimilates the dissimilar by a second science, which those who hit on the discovery have named _stereometry_ [the gauging of solids], a device of God's contriving which breeds amazement in those who fix their gaze on it and consider how universal nature molds form and type by the constant revolution of potency and its converse about the double in the various progressions. The first example of this ratio of the double in the advancing number series is that of 1 to 2; double of this is the ratio of their second powers [1:4], and double of this again the advance to the solid and tangible, as we proceed from 1 to 8 [1, 2, $2^2$, $2^3$]; the advance to a mean of the double, that mean which is equidistant from lesser and greater term [the arithmetical], or the other mean [the harmonic] which exceeds the one term and is itself exceeded by the other by the same fraction of the respective terms--these ratios of 3:2 and 4:3 will be found as means between 6 and 12--why, in the potency of the mean between these terms [6,12], with its double sense, we have a gift from the blessed choir of the Muses to which mankind owes the boon of the play of consonance and measure, with all they contribute to rhythm and melody. (990C - 991B).[187]

To understand van der Waerden's interpretation of this passage, we must consult the "arithmetical" Books, VII-IX, of the _Elements_. In Definition VII.21[188] of the _Elements_, similar plane and solid numbers are defined as being those with proportional sides. Thus the plane numbers _ab_ and _cd_ are similar if _a_:_c_ = _b_:_d_, while the same is true of the solid numbers _abc_ and _def_, if _a_:_d_ = _b_:_e_ = _c_:_f_. Now Propositions

VIII.18, [189] and VIII.20[190] amount to asserting that two plane numbers are similar if and only if a mean proportional (number) exists between them, while Propositions VIII.19[191] and VIII.21[192] amount to asserting that two solid numbers are similar if and only if two mean proportionals (numbers) lie between them.

Based on this, van der Waerden offers the following interpretation for what is meant by mensuration (γεομετρία) and stereometry (στερεομετρία).[193] A well-known and simple construction in plane geometry shows how, given any two lines (which may or may not represent numbers), one can construct their geometric mean (cf. EE, VI.13). By VIII.18 and 20, this mean will itself be a number if and only if the original lines, representing numbers, were similar. What was for a long time not known, however, was how, given any two lines, to construct two other lines in continuous proportion with the first two, i.e., how to insert two means between two given lines.[194] According to Proclus, Hippocrates of Chios had shown that solving this problem would yield a solution to the famous Delian problem of doubling a given cube, i.e., finding the side of a cube double in volume to a given cube.[195] Thus doubling the cube is equivalent to inserting two mean proportionals between given lines, which, by VIII.19 and 21, will yield means which are numbers if and only if the given lines represent similar solid numbers to begin with. Van der Waerden therefore, argues that:

> . . . solid geometry is defined [in the Epinomis, and hence by Plato] as "the new art, which teaches us how to make similar in this sense two numbers which, as given, are not similar." In particular therefore, solid geometry shows how to transform any number into a cube, and hence how to construct two cubes the ratio of whose volumes is equal to that of two arbitrary integers. For the Epinomis, this is evidently the problem of solid geometry; no other definition of this subject is given.[196]

This testimony he then couples with the following passages from Plato's Republic:

> [Geometricians] speak, as you doubtless know, in terms redolent of the workshop. As if they were engaged in action, and had no other aim in view in all their reasoning, they talk of squaring, applying, extending and the like, whereas, I presume, the real object of the whole science is knowledge. (527a)[197]
>
> After plane geometry, I said, we proceeded at once to solids in revolution, instead of taking solids in themselves;

> whereas after the second dimension the third, which is concerned with cubes and dimensions of depth, ought to have followed.
>
> That is true, Socrates; but so little seems to have been discovered as yet about these subjects. (528b)[198]

Van der Waerden's translation replaces "concerned with cubes and dimensions of depth" in the above passage by "the enlarging of cubes and everything that has depth," which appears to be due to a misreading of the word ἀύξην, which literally means "increase," but in this context the increase refers to dimension not size.[199]

Putting these two sources together, van der Waerden concludes:

> Confrontation of this definition [in the Epinois] with the passage in Plato's Republic, in which the purpose of solid geometry is defined as "the enlargement of cubes and of all things which have depth," shows that for Plato himself, the enlargement of a cube in a given ratio is also the outstanding problem of solid geometry. And now it also becomes clear [!] why he can write that these things do "not appear to have been investigated yet." Apparently the solution of Archytas had not yet been obtained, at least not yet known in Athens, around 375, when Plato was making his plans for The Republic. Perhaps Plato got the news of this solution just before the publication of The Republic, and was then led to the more optimistic tone noticeable at the end of the passage quoted above[200] [which, following his translation, reads: ". . . nevertheless, in the face of all these obstacles [these studies] force their way by their inherent charm and it would not surprise us if the truth about them were made apparent"].[201]

To this we can only say, along with the sentiment expressed in the final sentence of van der Waerden's quotation, "perhaps indeed." Surely this argument, based as it is on the flimsiest of evidence, should be taken for what it is, namely an interesting, bold, nay, wild conjecture that need not be taken very seriously. Certainly it is not the sort of stuff upon which one would want to erect a broad, sweeping theory about the character of Greek geometry! And yet this is precisely what van der Waerden goes on doing by spinning a wild fantasy based on his conviction that the problem of duplicating the cube practically defined the scope of solid geometry for the Greeks! Thus he continues:

> In the above passage, Plato mentioned as the most important planimetric operations, to change an area into a square (τετραγωνίζειν), the application and the addition of areas. Now, all these operations come from geometric algebra; changing an arbitrary rectilinear area $\underline{F}$ into a square amounts to solving the pure quadratic $\underline{x}^2 = \underline{F}$; the application of an area $\underline{F}$ to a line $\underline{a}$, without excess or deficiency, reduces to the solution of the linear equation $\underline{ax} = \underline{F}$; with excess or with deficiency, it leads to the solution of an arbitrary quadratic equation or to the solution of a system of 2 equations with 2 unknowns of the form
>
> $$\begin{cases} \underline{x} + \underline{y} = \underline{a} \\ \underline{xy} = \underline{F} \end{cases} \quad \text{or} \quad \begin{cases} \underline{x} - \underline{y} = \underline{a} \\ \underline{xy} = \underline{F} \,. \end{cases}$$
>
> Finally, the "adding" of areas or of lines is after all only the geometric equivalent of addition.[202]

But if we reexamine the passage of Plato, the conclusion that this is a statement about which operations are most important in plane geometry, would seem to be unwarranted on van der Waerden's part. In fact it appears to be no more than an offhand remark, the point of which has to do with the language geometers employ, not the nature of geometry. However, van der Waerden uses this remark to conclude:

> Thus we see, that what Plato calls plane geometry [!] is mainly the geometric algebra of the Pythagoreans.[203]

And then he states:

> It is not surprising therefore that he [Plato] looks upon solid geometry as the generalization of geometric algebra to space [!], i.e., as the geometric interpretation of the calculation with products of three factors each. The first new problem that arises here, is the solution of the pure cubic $\underline{x}^3 = \underline{V}$, i.e., the construction of a cube of given volume. It is therefore entirely logical to consider this as the central problem of solid geometry [!]. Obviously it is not the only problem; there are also equations of the form $x^2(\underline{x} + \underline{a}) = \underline{V}$ and other similar ones. That is why Plato adds the words "and

> everything which has depth" to "the enlargement of cubes," thus opening the way for other problems.
>
> For us, the most important result of this clarification is that we recognize more and more clearly the line of development of geometric algebra from the Babylonians, by way of the Pythagoreans, to the men of Plato's time.[204]

This sort of creatively manufactured evidence "is history-writing in its worst form."[205] What we ought to "recognize more and more clearly" is that to begin with, there are grave problems involved in the interpretation of much of Greek mathematics as being "algebraically" motivated. If Greek mathematics is algebraic it simply makes no sense. If nothing else, this is the inescapable conclusion of the preceeding pages. But if this be the case, then there is no longer any basis for accepting the theory that Babylonian mathematics, which, according to the communis opinio doctorum, was algebraic, was transmitted to ancient Greece and constituted the rational kernel of Greek mathematics.[206] On this issue we can do no better than listen to Szabó:

> Danach sieht also Neugebauers historische Konstruktion folgendermassen aus.
>
> Die Griechen haben zunächst eine Theorie vorgefunden, die 'babylonische Algebra', die für sie des Übernehmens gar nicht war, nachdem diese ihnen in ihren Schwierigkeiten nicht behilflich sein konnte. Darum haben sie diese 'als solche' auch nicht übernommen. Aber auf irgendeine ratselhafte Weise--wieso dies möglich war, erklärt uns Neugebauer nicht--haben die Griechen doch herausbekommen dass sie diese für sie, an sich gar nicht interessante 'babylonische Algebra' nur ins Geometrische übersetzen sollen, und sogleich erhalten sie darin ein nützliches Mittel, um ihre Schwierigkeiten der Inkommensurabilität zu überwinden. So entstand 'Euklids geometrische Algebra babylonischen Ursprungs'. Man wird wohl nicht behaupten, dass diese Konstruktion 'einfach' oder 'einleuchtend' wäre.[207]

It is in the same work that Szabó points out the similarities between "Babylonian algebra" and Diophantus's algebra, both of which operate with specific, rational numbers, unlike the so-called "geometrical algebra" of the Greeks.[208] Finally, summing up his criticism of the Neugebauer-van der Waerden views, Szabó states:

> "Warum haben die Griechen die babylonische Algebra nicht als solche übernommen, sondern geometrisch eingekeidet?" fragt van der Waerden in seiner Schilderung von Neugebauers Theorie. Ich glaube man konnte mit den selben Recht auch fragen: Wie hätten die Griechen die 'babylonische Algebra als solche' (ohne Geometrisierung!) übernehmen können? Was war überhaupt jene 'Algebra' die sie mit der Geometrisierung 'völlig unterdrückt' (?) hatten?
>
> Meiner Ansicht nach [and we fully concur] ist die ganze historische Theorie von der Übernahme der 'babylonischen Algebra' bzw. von ihrer Geometrisierung durch die Pythagoreer um kein Haar besser begründet, als die Datierung dieser Übernahme auf die Zeit 'nur um wenige Jahre' nach dem Fragment B 4 des Archytas.[209]

As for the Delian problem, there is, of course, ample evidence indicating that it was in fact the subject of immense fascination for Greek geometers.[210] There is no good reason to think, however, that the problem of doubling the cube had widespread implications for the subject of solid geometry as a whole (the Epinomis passage is a bad reason, and the excerpts from the Republic no reason at all), and there is even less reason to believe that "it arose from the translation of the Babylonian cubic equation $x^3 = V$ into spacial geometric algebra."[211]

2. There is also absolutely no reason to have to turn to sources like these in order to form an assessment of the character of Greek solid geometry--there are hundreds of pages of Greek stereometrical arguments extant, running the gamut from Archytas's solution to the Delian problem,[212] through Books XI-XIII of the Elements, down to Archimedes' works (e.g. On the Sphere and Cylinder).[213] In Book XI, for example, which undoubtedly contains some of the oldest material on the subject, we find clear parallels to the two-dimensional techniques employed in Books I and VI. (Compare, e.g., I.36 and XI.31,[214] or VI.1 and XI.32).[215] A further step is taken in Book XII where the "method of exhaustion" is introduced for the first time in the Elements,[216] making it possible to extend many of the results of Book XI to solids bounded by curvilinear surfaces, e.g., cones, cylinders, spheres, etc. (Compare XI.33[217] with XII.12[218] and XII.18).[219] Finally, in Book XIII, the Platonic solids

are inscribed in a given sphere and the lengths of each of their sides compared, using the scheme (for the case of the icosahedron and dodecahedron) established in Book X for classifying incomensurable magnitudes.[220] Certainly there is more than enough source material available here to enable one to formulate a reasonably sound overall conception of three-dimensional Greek geometry without having to rely on wild speculation about obscure and problematic passages whose principal point, moreover, has nothing to do with the specific character of Greek geometry at all!

If it appears to the reader that van der Waerden has gone way out of his way to advance this theory of his, it might then seem reasonable to ask why. The answer seems to be clear enough. For, if Greek geometers were doing "algebra in disguise" (whether Babylonian or not), there is no conceivable reason for them to have restricted this "algebra" to two dimensions, particularly since, as we shall soon see, almost everything that we find in two-dimensional "geometric algebra" can be extended quite effortlessly to dimension three! Of course, van der Waerden would probably only take this as a confirmation of the argument he has made all along, but this is hardly the case. None of the arguments we are about to present appears anywhere in the extant corpus of Greek mathematics, nor, we submit, is there anything even remotely similar. Moreover, many of these three-dimensional techniques can be employed to obtain simple (non-Greek) solutions to "algebraic" problems that evidently cost the Greeks a good deal of effort. We present these reasonings here as a kind of reductio ad absurdum meant to display peremptorily all that is wrong methodologically and historically with the arguments of the "geometrical algebraists."

In the last section, we showed that it was a simple matter to extend II.1 to three dimensions, and that, in fact, using "application of areas" one could obtain a geometric proof of the algebraic identity:

$a(B + C + D \ldots) = aB + aC + aD + \ldots,$

where B, C, D, etc. represent "products." One can also obtain three-dimensional versions of many other results in Book II, e.g. II.4,[221] which algebraically just says that $(a + b)^2 = a^2 + 2ab + b^2$. The corresponding 3-D result, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ can be proven using the diagram of II.4, (cf. Fig. IV.1) appropriately adapted to one dimension higher (Fig. IV.2). Thus instead of passing lines through the square, we

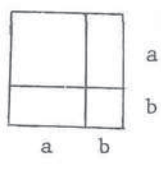


Fig. IV.1

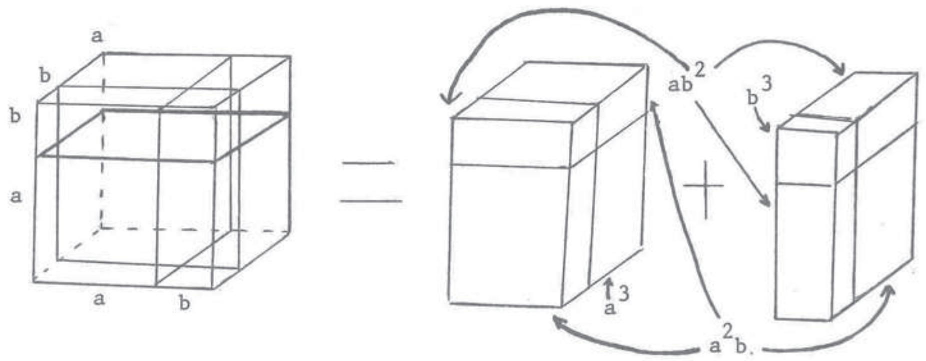



Fig. IV.2

pass three planes through the cube, dividing it into eight pieces. Adding them together gives the desired result! No comment.

Let us now go back and reconsider the problem of solving the general quadratic equation. The reader will recall that we had to struggle a good deal to interpret VI.28 and 29 as solutions to the pair of equations $ax \mp \frac{b}{c}x^2 = C$, and that once this was accomplished the alleged correspondence between the Euclidean proof and the algebraic technique of completing the square was, at best, superficial. In the course of that analysis we offered some "alternative" solutions in order to give just some idea of how far off the mark this whole interpretation really was.[222] What we would like to show now is that if we allow ourselves the freedom to do "geometric algebra" in three-dimensions, according to van der Waerden's injunctions, it is absolutely child's play to produce a solution for the general quadratic, a solution which the Greeks could not have missed, had they been doing "3-D geometrical algebra" à la van der Waerden!

Let us begin with the "usual" form for the quadratic, namely $ax^2 + bx = c$. When we encountered this equation earlier, we either had to ignore the fact that, in two dimensions, it makes no sense or else (following Heath, et al.) we had to rewrite it in the awkward form $\frac{p}{q}x^2 \pm rx = S$, which, as we saw, took a good bit of fancy stepping to unravel. But, if we allow ourselves the added "elbow-room" provided by 3-space, none of these difficulties arises at all, as there is a very natural interpretation for $ax^2 + bx = c$ in this setting, namely, let a, b, and c represent one, two, and three dimensional magnitudes respectively. In the 'geometric arithmetic," the paradigms one would employ for this purpose would be lines, rectangles, and rectangular solids (Fig. IV.3). To solve $ax^2 + bx = c$, apply I.45A to obtain $b = ap$, $c = ars$. Hence $ax^2 + apx = ars$ thus $a(x^2 + px) = ars$, by the 3-D version of II.1, so $x^2 + px = rs$ (Equals divided by equals are equal), and $rs = m^2$, by II.14, $\therefore\ x^2 + px = m^2$. And this is precisely the equation that we already solved earlier by means of II.11! Again, no comment.

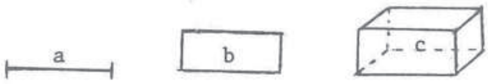



Fig. IV.3

This same argument can be used, without resorting to the third dimension, if we are willing to interpret the coefficients in the manner that Heath and the other geometrical algebraists do. Thus if $a = \frac{m}{n}$ is a ratio of lines, b a line, and c a rectilineal figure (hence $= d^2$, by II.14), the equation becomes $\frac{m}{n}x^2 + bx = d^2$. This is exactly the formulation given by Heath in connection with VI.29, only that now we apply a touch of "geometric arithmetic" instead of the "convoluted" argument in Euclid. We begin by applying I.45A to obtain $bx = my$ and $\frac{x^2}{n} = z$. Thus the equation becomes $mz + my = d^2$. Now use II.1 to write $mz + my = m(z + y)$,

$$m(z + y) = d^2,$$

and, dividing both sides by $m$,

$$z + y = \frac{d^2}{m}, \text{ and since } y = \frac{bx}{m}, \text{ we obtain}$$

$$\frac{x^2}{n} + \frac{bx}{m} = \frac{d^2}{m}.$$

Now multiply both sides by $n$, and then reapply II.1:

$$n\left(\frac{x^2}{n} + \frac{bx}{m}\right) = \frac{nd^2}{m}$$

$$\frac{nx^2}{n} + \frac{nbx}{m} = \frac{nd^2}{m}.$$

Finally, let $p = \frac{nb}{m}$ and $r^2 = \frac{nd^2}{m}$, by using I.45A and II.14 respectively. This yields

$x^2 + px = r^2$ which, as before, is solved using the

argument in II.11!

These arguments, we believe, conclusively show that the claim that propositions like VI.28, 29 were motivated by the desire to obtain a solution to the general quadratic equation is a historically empty claim. Evidently the Greek way of looking at these things was significantly different from our own, because any attempt to press the "analogy" between their approach and the technique used in an algebraic solution leads to a historical and mathematical dead end; moreover the willingness to use just a few light assumptions drawn from "geometric arithmetic" suffices to produce relatively simple solutions to quadratics that allow one to bypass entirely the proportion theory of Book VI. This too, then, is non-Greek. For surely those "god-like men of old" (Proclus's term for the Pythagoreans) would not have missed these easy solutions if they were interested in solving general equations involving the abstract notion of magnitude as number. In this regard, one should recall the passages in Plutarch that speak in such reverent terms about Proposition VI.25 of the Elements,[223] which, as we saw, was one of the two key ingredients in the proof of VI.29. Is it not strange, then, (if the Greeks were primarily interested in solving equations, that is) that it is VI.25, and not VI.28, 29, that is so lauded as a profound discovery? After all, the algebraic content in VI.25 is practically nil,[224] and yet it is esteemed as a result of transcendent metaphysical importance, its discovery being attributed to Pythagoras himself. But, of course, who would pose such a question seriously? It should be perfectly clear to the historically minded reader why VI.25 and not VI.29, is alluded to in Plutarch. It is our view that this is totally consistent with a balanced and sane view of Greek geometry, which points to the obvious fact that, from a geometric standpoint, VI.25 bears all the markings of a seminal result, whereas VI.29 is, comparatively speaking, only of incidental interest.

Now it might be objected (and rightfully so) that the 3-dimensional solution concocted above goes too far in the direction of pure magnitude, so that instead of offering a quasi-general "Greek-style" solution it really only strips the problem down to bare bones (lines, rectangles, and rectangular solids) thereby losing most of its general geometric content. What we will now show is that it is not at all difficult to develop the necessary tools in order to conclude that there is no loss of generality involved in taking $b$ and $c$ to be a rectangle and a rectangular solid respectively as in our solution above to $ax^2 + bx = c$. To do this, it suffices to develop the three-dimensional analogues for I.45A and II.14. The 3-D version of I.45A reads:

> To a given straight line to apply a rectangular solid equal to a given rectilineal solid. (Notice that we have taken the given angle in I.45A to be a right angle simply because, for "geometric arithmetic," no other angle matters.)

Now a "rectilineal solid figure," i.e., a polyhedron, can be "triangulated," which here means decomposed into solid triangles, or pyramids (Fig. IV.4). According to Archimedes, the volume of the pyramid was first discovered by Democritus (ca. 460 B.C.), who learned that the pyramid is one third part of the prism, having the same base and equal height.[225] (This result also appears as a Porism to Euclid XII.7.) Thus each of the pyramids that appears in the "triangulation" of the given rectilineal solid can be computed as a product: $V = \frac{1}{3} Bh = \frac{1}{6} abh$, as $B = \frac{ab}{2}$ (cf. Fig. IV.5). Using I.45A proper we can "apply" this "rectangular solid" to the given line p:

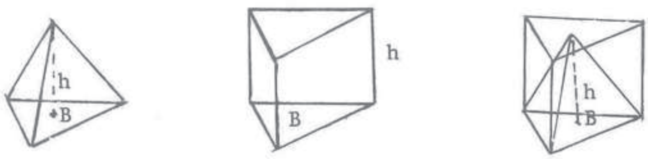



Fig. IV.4

$\frac{1}{6} abh = pqr$. Doing this for each of the pyramids we obtain: Rect. Fig. = $pq_1r_1 + pq_2r_2 + \ldots = p \cdot (q_1r_1 + q_2r_2 + \ldots)$, by our 3-D version of II.1. Now apply I.45A to each of the rectangles in parentheses to obtain $p \cdot (qs_1 + qs_2 + \ldots) = pq\,(s_1 + s_2 + \ldots)$, by II.1 proper. Adding the lines $s_1, + s_2 + \ldots = s$ yields the desired "rectangular solid" pqs.

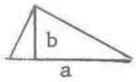



Fig. IV.5

The 3-D version of II.14 reads:

To construct a cube equal to a given rectilineal solid. To find this we simply apply our 3-D I.45A above to transform the given rectilineal solid to a rectangular solid $\underline{pqr}$. By II.14 proper, we can replace $\underline{pq}$ by $\underline{s}^2$ thus obtaining $\underline{rs}^2$. Now we can apply one of several techniques available to insert two mean proportionals between s and r. Thus,

$$s:x = x:y = y:r. \quad \text{By VI.16,}$$
$$x^2 = sy \quad \text{and} \quad xy = sr.$$

Hence $x^3 = sxy$ and $s^2r = pqr$. Thus we have rather easily found a cube equal to the given rectilineal solid.

Using these ready devices, we can now see that there is very little added difficulty in solving the equation $ax^2 + bx = c$, where $\underline{a}$ is a line, $\underline{b}$ an arbitrary plane rectilineal figure, and $\underline{c}$ an arbitrary rectilineal solid. For by using I.45A and its three-dimensional analogue, there is no loss in generality in taking $\underline{b}$ and $\underline{c}$ to be a rectangle and a rectangular solid as we did in our original solution. The 3-D version of II.14 gives, of course, an immediate solution to the equation $\underline{x}^3 = \underline{c}$, where $\underline{c}$ is an arbitrary rectilineal solid!

And the moral is: that ain't what the Greeks were doing.

## Conclusion

The tally is clear. The standard interpretation of the history of ancient Greek mathematics is wrong on all counts, not "just" historical, philosophical, and linguistic, but also, significantly enough, mathematical. Greek geometry, in general, and the problem of "application of areas," in particular, is not about solving equations, but rather about the study of the space in which we live and its properties, including its metrical properties, by means of *sui generis*, Greek geometric procedures which, for the Greek were not recudible to a simpler, arithmetic-algebraic *modus operandi*. Greek mathematics is not camouflaged algebra. It is essentially Greek geometry. The Greeks did not hide their algebraic line of reasoning behind the clumsy shroud of geometrical expression. There is nothing mysterious and unGreek lurking in hiding in the background of Greek geometry. In general, when a culture cannot say something it remains silent. This is true of the Greeks too. "Part of being ignorant of something is being ignorant of your ignorance. If you know that you are ignorant, your ignorance *stricto sensu* has disappeared. And the Greeks, clearly, did not know that they did not know algebra. So they did not hide their ignorance behind a geometrical screen"[226]

There is not one trace of authentic algebraic equations in classical Greek geometry. This includes, needless to say, Apollonius's Conics and Archimedes' ΠΕΡΙ ΕΛΙΚΩΝ, as well as Archimedes' other works with which, however, we could not deal in this study. We chose instead to focus on the Elements for rather obvious reasons, having to do with the central place it occupies in the history of Greek mathematics in particular and that of Western mathematics in general, its encyclopedic, all-inclusive elementary (fundamental) character that enabled it to become the prototype of all postulatory-deductive treatises in the West up to Newton's Philosophiae Naturalis Principia Mathematica and beyond, and because the author of the Elements, ὀζστοιχειώτη , was generally respectful of the tradition that he incorporated and systematized. This mathematical tradition is most certainly, essentially geometric, not algebraic. And so, since there are substantive differences between the geometric and the algebraic language and way of reasoning, introducing algebraic symbols and manipulations into Greek mathematics, transcribing geometrical propositions by means of algebraic equations, reverting to algebraic notations and transformations whenever the Greek way of doing things seems "awkward" and "cumbersome" to the modern mathematician amounts to an utterly ahistorical procedure in the exegesis of Greek mathematics and should, therefore, be avoided like anathema.

The important differences between the language of geometry and that of algebra should be kept in mind.[227] A specific language (in this case, Greek) is not an indifferent, neutral, purely abstract, universal mantle that can drape and accomodate equally well any conceptual content." To a greater or lesser degree every language offers its own reading of life."[228] To comprehend Greek mathematics, the historian must insinuate himself into the otherness that is the Greek mathematical universe of discourse. Any procedure that aims at understanding Greek mathematics qua Greek phenomenon cannot adopt a hermeneutics that bodies forth the pliant, flexible, supremely malleable and moldable, indefinite, apeiron-like, neutral slime of the symbolic, algebraic world order. Thoughts, including mathematical thoughts, are not born naked to be later clothed in any indefinite, impartial, neutral garb.

Though to understand is to translate, there are good and bad translations. It is possible to argue that what historians of mathematics of the ilk criticized in this essay have been doing in discussing ancient mathematical texts is not sympathetic translation (interpretation), but rather transmutation (perhaps transubstantiation), i.e., transposition of the original verbal, rhetorical text into a non-verbal sign system, in this case the symbolic, algebraic system.[229] Indeed, the vulgate of the typical modern manufacturer of historical studies in the domain of the history of mathematics has been algebra. It has been a most unhappy choice.

All interpretation is translation, but there are faithful translations (interpretations) and transmutations, which betray the original text by subjecting it to the tensions and potentialities of a foreign, non-verbal language system. A single example should suffice here. Proposition II.4 of the Elements (which we encountered before) reads:

> If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.[230]

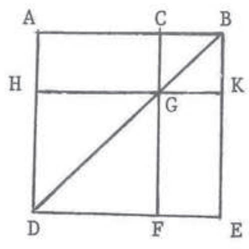
This has traditionally been transcribed (as we saw) by Heath, van der Waerden, and others as (1) $(a + b)^2 = a^2 + b^2 + 2ab$, and complete equivalence between the two expressions was claimed. This is, strictly speaking, not true. Once we have the formulaic transcription, we also have indefinitely many other algebraic identities obtainable from it by means of the standard rules of algebraic manipulation; for example,

$$(a + b)^2 - 2ab = a^2 + b^2$$

$$a + b = \sqrt{a^2 + b^2 + 2ab}$$

$$\frac{(a + b)^2 - b^2}{a} = a + 2b, \quad \text{etc., etc.}$$

These trivial consequences of (1) are not at all obvious and immediate (one could write some that would not even be obtainable) in the rhetorical, geometrical Euclidean enunciation. Add to this the fact that in Euclid, the proof of the claim is given by means of the transformations performed on a geometrical diagram, which in its completed form looks like the following figure,



and it becomes easy to see that there is a world of difference between the algebraic formulation, on the one hand, and Euclid's rhetorical enunciation and its geometrical proof, on the other hand. The former is supremely abstract and can undergo freely the innumerable manipulations of algebraic transformations, while the latter is inherently more limited by its verbality, syntax, and the spatial constraints of the Euclidean diagram accompanying and being an integral part of the proof. Ordinary language and two- or three-dimensional diagrams are not " completely equivalent" to their algebraic transmutations. Greek mathematical language and the modern algebraic language are not equivalent.

There is an important distinction, then, between intra-language translation and inter-language transmutation. The former is the acceptable historical hermeneutics, the latter is ahistorical by definition. Algebraic symbols are monosemic, while ordinary language constructs are typically polysemic. The letters of Euclidean proofs are not really symbols, they are rather proper names and thus both much richer and more circumscribed than the dry, exact, abstract algebraic symbols used to transcribe them and the proofs in which they are embodied. The Greek language is richer and, therefore, more ambiguous, mysterious, equivocal, and much less precise than the symbolism of algebra used to "translate" it.

It is fair to state that the mathematician's approach to the history of mathematics has traditionally been synchronic, horizontal, while what is clearly needed is a diachronic, vertical approach, whthout which it is not the _history_ of mathematics that is written but rather the _mathematics_ of history. If by _mathesis_ we mean the analytical reasonings represented by the new way of doing mathematics begun in the sixteenth and seventeenth centuries by (primarily) Viète, Descartes, and Fermat, the way which one of us called the "mos per symbola",[231] then, clearly, _Lexis non est mathesis_.

It has been argued that most contemporary historians of mathematics are Platonists in their approach.[232] They look in the past of mathematics for the eternally true, the unchanging, the constant. Also, culture was defined "topologically" as "a sequence of translations and transformations of constants."[233] This definition is obviously an _ex post facto_ construct of the interpreter of culture. Speaking of constants is possible only _after_ transformations were performed which, to the eye of hermeneutics, have left invariants. It makes no sense to speak of constants in the absence of change. And the historian is primarily interested in the event of change. What is preserved is important, but it can be assessed only in light of what has changed. To speak of constants, then, makes sense only because there are variables.

And historical assessment involves taking stock primarily of variability. If nothing changes there is no history.

Specifically, *after* algebra it is possible to find its rational kernel in geometry, thereby identifying, as it were, a constant; doing so, however, amounts to overlooking the very change that took place in the creation of algebra. This approach, therefore, that might sit well with the structural anthropologist is self-defeating (because blinding) for the historian. Victor Hugo is reported to have said: "Défense de déposer de la musique au long de cette poésie."[234] *Mutatis mutandis*, one can say, "Défense de déposer de l'algèbre au long de la géométrie grecque"!

We feel that in the preceding pages we have really put forward a truism, a rudimentary principle, but one whose crucial historiographic significance has traditionally been overlooked by those who have been writing the history of ancient mathematics. The damaging consequences of this oversight can hardly be overestimated. In any case, the reader who has accompanied us thus far will surely agree with us that the preceding constitutes an appropriate funeral for the variegated views embodied in the old and respectable historiographic label of "geometric algebra." *Fecimus quod potuimus, faciant meliora potentes*. And so, let us pray. יי נתן, יי לקח,
יהי שם יי מברך. κυριε ἐλέησον. Requiescat in pace.

NOTES

[95]An early example is Paul Tannery's 1882 study (Mémoires Scientifiques, Vol. 1 (1912), pp. 254-280), entitled "De la solution géométrique des problèmes du second degré avant Euclide."

[96]The Exact Sciences in Antiquity, p. 143.

[97]H. G. Zeuthen, Geschichte der Mathematik im Altertum und Mittelalter (Copenhagen: Andr. Fred Höst & Sön, 1896), p. 56.

[98]EE, vol. 3, p. 5.

[99]Proclus, A Commentary on the First Book of Euclid's Elements, tr. Glenn R. Morrow (Princeton: Princeton University Press, 1970), p. 332.

[100]Walter Burkert, Lore and Science in Ancient Pythagoreanism, tr. Edwin L. Minar, Jr. (Cambridge, Massachusetts: Harvard University Press, 1972), pp. 449-457, passim.

[101]Plutarch, Plutarch's Moralia in 15 volumes, vol. IX, tr. Edwin L. Minar, F. H. Sandbach, and W. C. Helmhold (Cambridge, Mass./London: Harvard University Press/William Heinemann Ltd., 1961), Table-Talk, VIII.2, 720A.

[102]EE, vol. 1, p. 344.

[103]Plutarch, Table-Talk, VIII.2, 720A and B. Cf. also Gregory Vlastos, Plato's Universe (Seattle: University of Washington Press, 1975), for a provocative discussion of the forces that influenced the composition of the Timaeus.

[104]Plutarch, Plutarch's Moralia, Vol. XIV, tr. Benedict Einarson and Phillip H. DeLacy (Cambridge, Mass./London: Harvard University Press/William Heinemann Ltd., 1967), A Pleasant Life Impossible, 1094 B and C. A more extended version of the story about Archimedes running from the bath is given by Vitruvius in De Architectura, for which cf. Thomas, SGM, Vol. 2, pp. 36-39.

[105]Heath in EE, vol. 1, pp. 372-73.

[106]Heath uses the term transformation of areas to denote those specific geometric operations which we have been calling "geometric arithmetic." Our term is meant to emphasize the role these operations supposedly play in the "geometric algebra," i.e., their content, rather than their geometric form. Cf. EE, vol. 1, pp. 346-47.

[107]The appropriate algebraic identities appear in Zeuthen (Die Lehre von den Kegelschnitten im Altertum, p. 12) and Heath (EE, vol. 1, pp. 372-73).

[108]Cf., n. 12 above.

[109]EE, vol. 1, p. 341.

[110]Ibid., p. 339.

[111]Proposition I.31: "Through a given point to draw a straight line parallel to a given straight line" (ibid., p. 315).

[112]Proposition I.29: "A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles" (ibid., p. 311).

[113]Postulate 5: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles" (ibid., p. 155). For a discussion of some of the history surrounding Postulate 5 cf. ibid., pp. 202-220.

[114]Proposition I.15: "If two straight lines cut one another, they make the vertical angles equal to one another" (ibid., p. 277).

[115]Ibid., p. 340.

[116]Ibid., p. 345.

[117]Definition I.19: "Rectilineal figures [Σχήματα ενθύγραμμα] are those which are contained by straight lines ..." (ibid., p. 187).

[118]Proclus, Commentary on Euclid, p. 335.

[119]EE, vol. 2, p. 253.

[120]Proposition VI.13: "To two given straight lines to find a mean proportional" (ibid., p. 216).

[121]Proposition VI.18: "On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure" (ibid., p. 229).

[122]Proposition VI.19: "Similar triangles are to one another in the duplicate ratio of the corresponding sides" (ibid., p. 232). Porism: "From this it is manifest that, if three straight lines be proportional, then, as the first is to the third, so is the figure described on the first to that which is similar and similarly described on the second" (<u>ibid</u>., p. 233). As Heath points out (<u>ibid</u>., p. 234), the porism is really out of place here and should follow VI.20 instead.

[123]Proposition VI.1: "Triangles and parallelograms which are under the same height are to one another as their bases" (<u>ibid</u>., p. 191).

[124]This should be an immediate inference from V.14: "If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third, the second will also be greater than the fourth; if equal, equal; and if less, less." Instead, the argument in the <u>Elements</u> uses VI.16, to consider the proportion <u>alternado</u> (ἐναλλάξ), thus:

$$\therefore (\triangle ABC) : (\square BE) = (\triangle KGH) : (\square EF)$$

in order to conclude that since the first equals the second, so must the third equal the fourth. Yet no such proposition exists in Euclid. This is only one of many difficulties with the received proof of VI.25. <u>Cf</u>. <u>ibid</u>., pp. 254-55, and the following note , here.

[125]If ABC is replaced by the rectilineal figure T, then I.45A can be applied to obtain BE as before. When, after constructing GH, it becomes necessary to construct T' on GH similar to T, we must use VI.18 in its <u>full</u> generality. This requires that the rough spots in the proof of VI.18 be smoothed out, for which see Heath's recommendations in <u>EE</u>, vol. 2, pp. 230-32. The rest of the proof then goes through smoothly by placing the Porism to VI.19 after VI.20 and by utilizing V.14 rather than V.16 as found in Euclid.

[126]<u>Ibid</u>., p. 258.

[127]Although VI.28 is utilized (at least implicitly) in several places throughout Book X (X.17, 18, 33-35, 39-41, 54, 55, 57, 58, 76-78, 91-96), all of these instances require only the case where the "defect" is a square, and this case can be proven with little difficulty using only the techniques of Book II (<u>cf</u>. Simson's solution in <u>EE</u>, vol. 1, pp. 383-84). Moreover, VI.28 is the only result of the three (VI. 27-29) utilized in Book X, as there is never any need for an "application

with excess" (VI.29). Thus, although we acknowledge the great importance in Book X of "application of areas" in general, we cannot agree with Heath that these three propositions in particular "constitute the foundation of Book X of the Elements."

[128] EE, vol. 2, p. 258.

[129] Ibid., p. 265.

[130] Cf., n. 121 above

[131] Proposition VI.21: "Figures which are similar to the same rectilineal figure are also similar to one another" (EE, vol. 2, p. 239).

[132] Proposition VI.26: "If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, it is about the same diameter with the whole" (ibid., p. 255).

[133] Proposition I.36: "Parallelograms which are on equal bases and in the same parallels are equal to one another" (EE, vol. 1, p. 331).

[134] Proposition I.43: "In any parallelogram the complements of the parallelograms about the diameter are equal to one another" (ibid., p. 340).

[135] Proposition VI.24: "In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another" (EE, vol. II, 251).

[136] EE, vol. 1, p. 349.

[137] Proposition I.41: "If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle" (ibid., p. 338).

[138] Proposition I.4: "If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend" (ibid., p. 247).

[139] "If we listen to those who like to record antiquities, we shall find them attributing this theorem [I.47] to Pythagoras and saying that he sacrificed an ox on its discovery. For my part, though I marvel at those who first noted the truth of this theorem, I admire more the author of the Elements, not only for the very lucid proof by which he made it fast [χατεδήσατο], but also because in the sixth

book he laid hold of a theorem even more general than this and secured it by irrefutable scientific arguments. For in that book he proves generally that in right-angled triangles the figure on the side that subtends the right angle is equal to the similar and similarly drawn figures on the sides that contain the right angle" (Proclus, Commentary on Euclid, pp. 337-338).

[140] Cf. Thomas, SGM, Vol. 1, p. 181.

[141] EE, vol. 1, p. 382.

[142] Ibid., p. 385.

[143] The fact is that throughout Book II and in most applications of Book II (e.g., in Book X) one finds this emphasis on rectangle formation rather than on the formed rectangles themselves. This suggests, of course, that there was an operational significance that the Greeks associated with this particular geometric construction. But while this seems indeed to be the case, it does not follow that this operation was the cornerstone of a "geometric algebra," unless one takes the word "algebra" to mean any system whatsoever that manipulates mathematical entities with sufficient facility so as to make it possible somehow to detect general abstract relationships. For more on this point cf. n. 12 above.

[144] Cf. n. 134 above.

[145] Cf., for example, van der Waerden, SA, pp. 120-21.

[146] EE, vol. 1, p. 402.

[147] Proposition I.46: "On a given straight line to describe a square" (ibid., p. 347).

[148] Ibid., p. 409.

[149] Ibid., p. 410.

[150] EE, vol. 2, p. 216.

[151] The precise statement in Aristotle reads as follows: "Since what is clear or logically more evident emerges from what in itself is confused but more observable by us, we must reconsider our results from this point of view. For it is not enough for a definitive formula to express as most now do the mere fact; it must include and exhibit the ground also. At present definitions are given in a form analogous to the conclusion of a syllogism; e.g., What is squaring? The

construction of an equilateral rectangle equal to a given oblong rectangle. Such a definition is in form equivalent to a conclusion [i.e., it has nothing in it corresponding to a middle term (Smith)]. One that tells us that squaring is the discovery of a line which is a mean proportional between the two unequal sides of the given rectangle discloses the ground of what is defined" (The Works of Aristotle, ed. W. D. Ross, Vol. III., De Anima, tr. J. A. Smith (Oxford: Clarendon Press, 1931; 1963 Reprint) II.2 413a 11-19). This testimony of Aristotle adds weight to our contention that this result was primarily significant for Greek proportion theory and has nothing whatsoever to do with solving a pure quadratic equation. Heiberg and Heath both apparently agree that II.14 was probably proven by means of proportion theory prior to Euclid's time (cf. EE, vol. 1, p. 410). It does not seem unreasonable to conjecture that the proof of II.14 given in the Elements is Euclid's own, and that he meant it to be an elegant capstone to Book II while, at the same time, demonstrating the power of the results that he had obtained up to then. The three ingredients of "application of areas" (I.45), "gnomon-relationships" (II.5), and the "Pythagorean theorem" (I.47), all of which blend together so beautifully here, make this a fitting climax to Book II that would have, no doubt, greatly appealed to Euclid's aesthetic sensibilities, as well as to those of his contemporaries.

[152]The claim of II.6A, by which we mean the generalized II.6, can best be understood by referring to an appropriate diagram. Thus, compare Figs. III.7 and III.10. Using Fig. III.7 one translates II.6 as AM + EH = ED, while the analogous reading for II.6A from Fig. III.10 yields AO + FB = FO. We will not attempt, for obvious reasons, to render the statement of II.6A in the language of Greek geometry.

[153]Cf. n. 133 above.

[154]Cf. n. 134 above.

[155]One will find the details and other related constructions in B. L. van der Waerden, "Defence of a 'Shocking' Point of View," pp. 207-209.

[156]EE, vol. 2, p. 260.

[157]The discovery of diorismi is attributed by Proclus (probably drawing on Eudemus) to one Leon, the pubil of Neoclides, about whom we know next to nothing (cf. Thomas, SGM, vol. 1, pp. 150-51).

[158]EE, vol. 2, p. 257.

[159]These four results are then used to prove several more in Book X, e.g., X.35, X.39-41, X.54-59, X.91-92, etc. (cf. EE, vol. 3, pp. 1-259).

[160]Cf. Thomas, SGM, vol. 2, pp. 304-323.

[161]Cf. EE, vol. 1. pp. 387-88. Simson also derives a simple solution for the equation corresponding to the application with square "defect," i.e., $ax - x^2 = b^2$ (cf., ibid., p. 384).

[162]This is not the case, however, with van der Waerden's work (cf. for example, SA, pp. 118-126, 138-41, 265-66).

[163]EE, vol. 1, p. 343.

[164]The "soft evidence" is discussed in section I above.

[165]Proposition II.1: "If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments" (EE, vol. 1, p. 375). According to Heath (ibid., p. 376), this proposition is "... the geometrical equivalent of the algebraical formula $a(b + c + d + \ldots) = ab + ac + ad + \ldots$," and it is obvious that with such an approach an analogous relation holds for rectangular solids. This is, to judge from the written record, a rather innocent step that the "geometrical algebraist" would take without qualms. Thus one obtains the identity $a(B + C + D + \ldots) = aB + aC + aD + \ldots$, where the capital letters represent rectangles. Here is the simple and convincing proof: Apply I.45A to write $B = pb$, $C = pc$, $D = pd$, etc., whence $a(B + C + D + \ldots) = a(pb + pc + pd + \ldots)$. By II.1, $pb + pc + pd + \ldots = p(b + c + d + \ldots)$, so $a(pb + pc + pd + \ldots) = ap(b + c + d + \ldots) = apb + apc + apd + \ldots$ from the obvious geometric fact involved, namely, that if one side of a rectangular solid is cut into any number of pieces, the smaller solids so obtained will, added together, equal the original. And, since $apb + apc + apd + \ldots = aB + aC + aD + \ldots$, it follows that, $a(B + C + D + \ldots) = aB + aC + aD + \ldots$, q.e.d.; another bright victory for the cause of "geometrical algebra."

[166]Van der Waerden ("Defence of a 'Shocking' Point of View," p. 206), following the lead of E. Neuenschwander, thinks that the term "parallelogram" was not introduced in Greek geometry until the time fo Eudoxus (early 4th century B.C.).

Thus "when the Pythagoreans invented their application of areas with defect or excess, the defect or excess was probably required to be just a square, not a parallelogram similar to a given one." Now leaving aside the imposing evidence against the claim that "application of areas" way a Pythagorean discovery, there seems to be some peculiar reasoning involved here. For as we have seen, van der Waerden (cf. SA, pp. 121-123), Heath, et al. interpret VI.28, 29 as quadratic equations essentially be stripping them of their geometric content. The given rectilineal figure C plays no role other than to represent a given two-dimensional magnitude, while the given parallelogram D is taken (by Heath) to be a rectangle. By doing this, these practitioners of "geometric algebra" severely restrict the actual, given situation in VI.28, 29 in order to produce "nice" equations: and now van der Waerden asks us to believe that the original "Pythagorean" invention was not even this! Rather, as he says, "the defect or excess was probably required to be just a square ..." But on what grounds? Even, if we grant that the term "parallelogram" was of post-Pythagorean vintage, why does that rule out having an "excess" or "defect" similar to a given rectangle? It would seem that the passage form Plato's Meno (86E-87B) where the defect seems to be only a rectangle and not a square, would lead one to assume that this "ancient method," whether Pythagorean or not, was certainly known in greater generality than van der Waerden's remark is meant to imply. His interpretation paints a rather pretty picture: II.5 and II.6 are indeed just what is needed in order to solve the problems in "application of areas" with square "excess" and "defect," but there seems to be no good evidence to support van der Waerden's view other than the desire to obtain a pretty picture.

[167] Proposition VI.12: "To three given straight lines to fina a fourth proportional" (EE, vol. 2, p. 215).

[168] This, of course, says nothing earth-shaking algebraically, however, geometrically it is an altogether different matter. Interpreted geometrically $\frac{bx}{c}$ is "division" à la Heath, whereas $\frac{b}{c}x$ is found by solving for the fourth proportional, using VI.12. To show that they are the same involves the following observation: Since Rect. $(b,x)$ = Rect. $(c, \frac{bx}{c})$, $\therefore b:c = \frac{bx}{c} : x$, by VI.16. On the other hand, $b:c = \frac{b}{c}x:x$ by definition of $\frac{b}{c}x$. It follows from V.11 (Ratios which are the same with the same ratio are also the same with one another" (EE, vol. 2, p. 158)) that $\frac{bx}{c} : x = \frac{b}{c}x : x$. Finally, by V.14, we conclude that $\frac{bx}{c} = \frac{b}{c}x$.

[169]This requires knowing that $(\frac{b}{c}x)x = \frac{b}{c}x^2$, where the latter term involves finding the fourth proportional $y$ in $b:c = y:x^2$. This is "easy" to show, for by using VI.12 one can find z such that $b:c = z:y$. Since VI.1 says that Rect. $(z,x)$: Rect. $(x,x) = z:x$, it follows in the "geometric arithmetic" that $b:c = z\cdot x:x^2$. Now $z = \frac{b}{c}x$, by definition, while $z\cdot x = \frac{b}{c}x^2$ for the same reason. Thus, hocus-pocus-preparatus, $z\cdot x = (\frac{b}{c}x)x = \frac{b}{c}x^2$ as required.

[170]Quoted in EE, vol. 2, p. 133.

[171]Cf. no 167 above.

[172]Cf. n. 123 above.

[173]In V.18 ("If magnitudes be proportional separando, they will also be proportional componendo" (EE, vol. II, p. 169)), Euclid assumes in the course of the proof that if three magnitudes be given, at least two of which are of the same species, then there exists a fourth proportional. However, this is never proven anywhere in the Elements, and its assumption here was, therefore, undoubtedly a slip on his part. The proof we have given can be extended without much difficulty to the case where one magnitude is an arbitrary rectilineal figure and not just a square (cf. EE, vol. 2, pp. 170-174). The inference we made by using VI.1 occurs explicitly in Book X as a lemma to X.22 (cf. EE, vol. 3, pp. 50-51).

[174]This effortless interplay between ratio and magnitude that Heath and others think constitutes Greek "geometric arithmetic" (cf. EE, vol. 2, p. 187) leads, as we shall show, to gross distortions when applied consistently.

[175]Van der Waerden argues, on rather frail evidence, contra this, namely, that transformation of volumes was indeed an important ingredient in Greek mathematics (SA, pp. 138-141). We shall have more to say about this matter shortly.

[176]The only evidence that Theaetetus ever extended the study of incommensurable magnitudes to the case of the so-called cubic irrationals is the phrase: "And similarly in the case of solids" at the end of a well-known passage (147D-148B) from Plato's Theaetetus (cf. Thomas, SGM, vol. 1, pp. 380-83). Yet van der Waerden, with characteristic boldness, writes that "he [Theaetetus] was able to extend the entire theory without difficulty ('like a stream of oil that flows without a sound') to the sides of commensurable cubes" (SA, p. 168). We are not familiar with the euphonious phrase that van der Waerden employs here, but "without a sound" aptly

describes what the ancient sources have to say on this matter. A more lengthy modern, and rather speculative discussion of this issue can be found in W. R. Knorr, The Evolution of the Euclidean Elements (Dordrecht-Holland/Boston-U.S.A.: Reidel, 1975), pp. 87,93,302.

[177]Thomas, SGM, vol. 1, p. 211.

[178]EE, vol. 2, p. 263.

[179]Greek practice would allow for the "multiplication" of both sides by a numerical ratio, however.

[180]This argument is adapted from Heath's, given in EE, vol. 2, pp. 266-67.

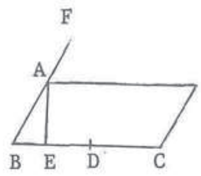
[181]Heath's remark in EE, vol. 1, p. 410.

[182]Proposition II.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments" (ibid., p. 379). The algebraic rendering is, of course, $(a + b)^2 = a^2 + b^2 + 2ab$.

[183]This is a convenient point at which to mention Euclid's Data, as it is regarded by some as being of "great importance for the history of algebra" (van der Waerden, SA, p. 198) and a "textbook on solving equations" (Freudenthal, "What is Algebra and What has it been in History?'" p. 194). In truth, there are relatively few propositions in the Data that are amenable to algebraic transcription, and most that are, e.g., Propositions 10-21, are rather trivial. There are four propositions (58, 59, 84, and 85) that transcribed lead to quadratics similar to those derived from VI,28 and 29. Proposition 86 is rendered by van der Waerden as the system:

$$\begin{cases} xy = F \\ y^2 = ax^2 + C \end{cases}$$

which he proceeds to solve by "introducing a new unknown z" (cf. SA, pp. 198-199). After obtaining an algebraic solution he writes: "All this is of course formulated in geometrical language; instead of y, z and y-z, Euclid writes AΔ, AE and EΔ, etc. But I have made no other changes in his reasoning" (ibid., p. 199). If we turn to the original (Data, ed. R. Simson, pp. 442-43), however, we find that van der Waerden has arbitrarily restricted it to the case where the "given parallelogram in a given angle" is a rectangle. Obviously, this makes it easier to get the




equation xy = F, yet even more serious is the damage this does to the proof. For Euclid's central argument involves dropping a perpendicular AE to BC, and utilizing triangle ABE in conjunction with BF (where SQ (BF) = Rect. (BC,CD)) which also never appears in van der Waerden's argument. Thus, far from having "made no other changes [other than notation] in his [Euclid's] reasoning," van der Waerden's version overlooks the main point of the proof. We shall not take up here the important question as to whether dramatic changes in language can be undertaken matter-of-factly without thereby necessarily creating changes in reasoning. On this issue see S. Unguru, "On the need" and "History of Ancient Mathematics: Some Reflections on the State of the Art."

[184] Cf. the proof of VI.25 in section III, no. 4 above, together with nn. 122, 124, and 125.

[185] Proclus gives an extensive discussion on the nature of elements, explaining that they serve the same function as do letters of the alphabet in the formation of language, and indeed the same word, στοιχεια, means letters in Greek. Cf., Proclus, Commentary on Euclid, pp. 59-61.

[186] SA, p. 139. In truth, there is very little consensus of opinion on that issue, but, according to Wilbur Knorr, (The Evolution of the Euclidean Elements, p. 93), the balance of opinion favors the view, pace van der Waerden, that the author was Plato's disciple, Philip of Opus.

[187] The Collected Dialogues of Plato Including the Letters, ed. Edith Hamilton and Huntington Cairns (New York: Pantheon Books, 3rd Printing, 1964), Epinomis, tr. A. E. Taylor, pp. 1531-32. Several writers have concocted speculative theories to explain this passage including Taylor himself (cf. A. E. Taylor, Plato/The Man and His Work, 5th ed. (London: Methuen and Co. Ltd., 1948), pp. 503-516, and the 1926 essay "Forms and Numbers: A study in Platonic Metaphysics" in A. E. Taylor, Philosophical Studies (Freeport, New York: Books for Libraries Press, a 1968 reprint of the 1934 edition), pp. 91-150.

[188] EE, vol. 2, p. 278.

[189]Proposition VIII.18: "Between two similar plane numbers there is one mean proportional number; and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side" (ibid., p. 371).

[190]Proposition VIII.20: "If one mean proportional number falls between two numbers, the numbers will be similar plane numbers" (ibid., p. 375).

[191]Proposition VIII.19: "Between two similar solid numbers there fall two mean proportional numbers; and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side" (ibid., p. 373).

[192]Proposition VIII.21: "If two mean propositional numbers fall between two numbers, the numbers are similar solid numbers" (ibid., p. 377).

[193]Cf. SA, p. 140.

[194]Thus given two lines a and b, one must find two other lines x and y such that: $a:x = x:y = y:b$.

[195]SGM, vol. 1, pp. 252-53.

[196]SA, p. 140.

[197]The Dialogues of Plato, tr. B. Jowett, 4 vols. (Oxford: Clarendon Press, 1969 reprint of the 4th ed. of 1953) Vol. II, p. 391.

[198]Ibid., p. 392.

[199]SA, p. 138. W. H. D. Rouse (Great Dialogues of Plato, tr. Rouse (New York: New American Library, 1956), p. 327, n. 1) writes: "ἀύξην, increase. We should say 'dimension.' 'Third increase' or 'cubic increase' meant to the Greeks the change of plane squares into solid cubes; 'forms having depth' refers to solids other than cubes."

[200]SA, p. 140.

[201]Ibid., p. 138.

[202]Ibid., pp. 140-41.

[203]Ibid., p. 141.

[204]...

[205]Van der Waerden's phrase ("Defense of a 'Shocking' Point of View," p. 200).

[206]Neugebauer, the originator of this theory and van der Waerden, its appropriator, are representative instances both of the view that Babylonian mathematics was algebraic in character and that Babylonian algebra was taken over by the Greeks (it is not at all clear *why* since it could *not* assist the Greeks in their "problems" with incommensurability), geometricized, and transformed into "geometrical algebra."

[207]*Op*. cit., pp. 12-13.

[208]*Ibid*., p. 19.

[209]*Ibid*., p. 20.

[210]For a survey of Greek activity devoted to the problem, *cf*. T. L. Heath, *A History of Greek Mathematics*, 2 vols. (Oxford: Clarendon Press, 1921), vol. 1, pp. 244-270; this work will hereinafter be refered to as *HGM*.

[211]*SA*, p. 161.

[212]*Cf*., *SGM*, vol. 1, pp. 284-289.

[213]*Cf*., E. J. Dijksterhuis, *Archimedes* in *Acta Historica Scientiarum Naturalium et Medicinalium*, vol. 12(1956), *On the Sphere and Cylinder*, pp. 141-221.

[214]Proposition I.36: "Parallelograms which are on equal bases and in the same parallels are equal to one another" (*EE*, vol. 1, p. 331). Proposition XI.31: "Parallelepipedal solids which are on equal bases and of the same height are equal to one another" (*EE*, vol. 3, p. 337).

[215]Proposition VI.1: "Triangles and parallelograms which are under the same height are to one another as their bases" (*EE*, vol. 2, p. 191). Proposition XI.32: "Parallelepipedal solids which are of the same height are to one another as their bases" (*EE*, vol. 3, p. 341).

[216]The key tool that is utilized in the "exhaustion" arguments that appear in the *Elements* is Proposition X.1 which is proved by means of Definition V.4 ("Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another" (*EE*, vol. 2, p. 114)) which, in turn, is equivalent to the so called "Axiom of Archimedes" (*cf*. *HGM*, vol. 1, pp. 326-329).

[217]Proposition XI.33: "Similar parallelepipedal solids are to one another in

the triplicate ratio of their corresponding sides" (EE, vol. 3, p. 342).

[218]Proposition XII.12: "Similar cones and cylinders are to one another in the triplicate ratio of the diameters in their bases" (ibid., p. 410).

[219]Proposition XII.18: "Spheres are to one another in the triplicate ratio of their respective diameters" (ibid., p. 434).

[220]Ibid., pp. 467-507.

[221]Cf., n. 182.

[222]Cf. section III, no. 6-8.

[223]Cf., section III, no. 2.

[224]The notion of similarity between geometric figures has no real counterpart in "geometric algebra" because algebra is concerned solely with content (i.e., size) and not with form (i.e., shape). For this reason, there is no "equation" derivable from VI.25, and, hence, there can be no question but that it was a purely geometric result.

[225]SGM, vol. 1, pp. 228-231.

[226]"On the Need," p. 107, n. 122.

[227]Ibid., pp. 75-77, passim.

[228]Steiner, After Babel, p. 473.

[229]We are using here Pierce's and Roman Jakobson's triadic theory of signs and meaning, which discerns between rewording, translation proper, and transmutation. Cf. After Babel, pp. 260-61.

[230]EE, vol. 1, p. 379.

[231]"On the Need," p. 111, n. 138.

[232]"History of Ancient Mathematics," Isis, vol. 70 (1979), p. 555.

[233]After Babel, p. 426.

[234]Quoted in René Berthelot, "Défense de la poésie chantée," La Revue Musicale, vol. 186 (1938), p. 90.