

## On 2-Paracontact Pseudo-Riemannian Manifolds with Minkowski Index

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### INTRODUCTION

Pseudo-Riemannian manifolds  $(M, g)$  with *Minkowski index* (abbr. PRMI) have been studied the first time in [23]. Such manifolds are by definition of even dimension, and in case  $\dim M = 4$ ,  $(M, g)$  is a general space-time. Let then  $(M, g)$  be a  $(2m + 2)$ -dimensional (PRMI)-manifold and let  $T_p(M)$  be the tangent space to  $M$  at  $p \in M$ . One may split  $T_p(M)$  as  $T_p(M) = H_p(M) \oplus S_p(M)$  where  $H_p$  is a  $2m$ -dimensional para-Hermitian vector space [14] and  $S_p$  a 2-dimensional spatial vector space.

On the other hand in last years several papers have been concerned with *almost  $r$ -contact* or  *$r$ -paracontact* structures for any value of  $r$  even or odd, (see for instance [3],[9]). Therefore in the present paper we assume that  $S_p$  carries a *2-paracontact structure* (denoted by 2-PC) defined by two structure vector fields  $\xi_r$  ( $r = m + 1, 2m + 2$ ) and we shall set  $S_p = D_C$ . It is proved that in this case, such a (PRMI)-manifold can be viewed as a Riemannian product  $M = M_h \times M_C$  where:

- (i)  $M_C$  is a *totally geodesic flat surface* tangent to the 2-paracontact distribution  $D_C$ ;
- (ii)  $M_h$  is a *minimal totally umbilical 2-codimensional submanifold* of  $M$ , tangent to the para-Hermitian distribution  $H_p$ .

Moreover the *bicontact vector field*  $\xi = \sum f_r \xi_r, f_r \in C^\infty M$  [9], defines an *infinitesimal homothety* on  $M$  and  $\text{Ric}(\xi) = -m \|\xi\|^2$ . Further the dual vector field  $\bar{\xi}$  of  $\xi$  with respect to the volume element of  $M_C$  is a *Killing vector field*.

In the 3<sup>th</sup> section we assume that  $M_h$  is structured by a *self-orthogonal foliate* connection (abbr. SO) defined by a global vector field  $V$ . Such connections have been already considered, when dealing with para-Hermitian or Hermitian manifolds (see for instance: [24], [20], [10]). A SO-foliate connection induces on  $M$  a *conformal 2-cosymplectic structure* defined by a 2-form  $\Omega$  such that  $d\Omega = 2(\nu + \eta) \wedge \Omega$ , where  $\eta$  and  $\nu$  are the dual forms of  $\xi$  and  $V$ , respectively. In this case  $M$  may be viewed as the Riemannian product  $M = M_I \times M_I^*$ , where  $M_I$  and  $M_I^*$  are  $(m + 1)$ -dimensional *coisotropic submanifolds* of defect  $d = m$ .

Making use of the *paracomplex operator*  $J$  of P. Libermann [14], it is seen that  $M_I$  and  $M_I^*$  may also be regarded as *antiinvariant* submanifolds of  $M$ . That is, one has  $JT_{p_I}(M_I) = T_{p_I}^\perp(M_I)$ . (see [28]).

Furthermore,  $V$  is a *null* (real) vector field and it defines:

- i) An *infinitesimal relative conformal transformation* of  $\Omega$ , i.e.  $d(\mathcal{L}_V\Omega) = \nu \wedge \mathcal{L}_V\Omega$ ;
- ii) An *infinitesimal automorphism* of all the  $(2q + 1)$ -forms  $\bar{v}_q$ , i.e.  $\mathcal{L}_V\bar{v}_q = 0$ , where  $\bar{v}_q = \bar{v} \wedge \Omega^q$ ,  $\bar{v} = \mu V = i_V\Omega$ .

On the other hand, the vector field  $JV$  commutes with  $V$ ; and it defines an infinitesimal automorphism of  $\Omega$ .

Finally, making use of the *complex vectorial formalism* (abbr. CVF) the following striking property emerges:

Any 4-dimensional (PRMI)-manifold endowed with a 2-paracontact spatial structure is a *conformal recurrent space-time*. Such a space-time is of Schwarzschild type and is endowed with a local symplectic structure [27].

## 1 PRELIMINARIES

Let  $(M, g)$  be a Riemannian or Pseudo-Riemannian  $C^\infty$ -manifold and let  $\nabla$  be the covariant differential operator defined by the metric tensor  $g$ . We assume that  $M$  is orientable and that the connection  $\nabla$  is symmetric.

Let  $\Gamma(TM) = \mathfrak{X}M$  and  $\flat : TM \rightarrow T^*M$  be the set of the sections of the tangent bundle  $TM$  and the *musical isomorphism* [19] defined by  $g$ , respectively. Next, following [19] we set  $A^q(M, TM) = \Gamma\text{Hom}(\wedge^q TM, TM)$  and notice that elements of  $A^q(M, TM)$  are vector valued  $q$ -forms,  $q \leq \dim M$ . Denote by  $d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM)$  the *exterior covariant derivative operator* with respect to  $\nabla$ . It should be noticed that generally  $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$ , unlike  $d^2 = d \circ d = 0$ . If  $p \in M$  then the vector valued 1-form

$dp \in A^1(M, TM)$  is the canonical vector valued 1-form of  $M$ . Since  $\nabla$  is symmetric, one has  $d^\nabla(dp) = 0$ . The operator

(1.1)  $d^\omega = d + e(\omega)$ , acting on  $\Lambda M$ , where  $e(\omega)$  means the exterior product by the closed 1-form  $\omega \in \Lambda^1 M$ , is called the *cohomology operator* [11].

Let  $(M, g)$  be a general space-time and  $S = \text{vect}\{h_A, A = 1, \dots, 4\}$  a vectorial basis. The elements of  $S$  determine what is called in Relativity a *null tetrad*, where the null vectors  $h_1, h_4$  are real whilst  $h_2, h_3$  are complex conjugate. If  $S^* = \{\theta^A\}$  denotes the coframe of  $S$  then the soldering form  $dp$  of  $M$  is expressed by  $dp = \theta^A \otimes h_A$  and since  $g(h_1, h_4) = 1, g(h_2, h_3) = -1$  the metric tensor  $g$  is expressed by

$$(1.2) \quad g = \langle dp, dp \rangle = 2(\theta^1 \otimes \theta^4 - \theta^2 \otimes \theta^3).$$

The complex vectorial formalism (abbr. CVF) was developed in [5] and it is based on the local isomorphism  $\Upsilon: L(4) \rightarrow SO^3(\mathbb{C})$ , where  $L(4)$  is the 4-dimensional linear group, and  $SO^3(\mathbb{C})$  is the 3-dimensional complex rotation group, respectively. The space of the 2-forms  $\theta^A \wedge \theta^B$  is isomorphic to the space self-dual 2-forms  $Z^\alpha (\alpha = 1, 2, 3)$ . These forms set up a basis of the complex space  $\mathbb{C}^3$  and are expressed by

(1.3)  $Z^1 = \theta^3 \wedge \theta^4, Z^2 = \theta^1 \wedge \theta^2, Z^3 = \frac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3)$ . Notice that  $Z^3$  is exchangeable with the metric tensor  $g$  and we agree to call it the *local almost symplectic form* of  $M$  (see [27]).

The first group of structure equations in  $\mathbb{C}^3$  is expressed by

$$(1.4) \quad \begin{cases} dZ^1 = \frac{1}{2}G_3 \wedge Z^1 - G_2 \wedge Z^3 \\ dZ^2 = \frac{1}{2}G_3 \wedge Z^2 + G_1 \wedge Z^3 \\ dZ^3 = \frac{1}{2}G_1 \wedge Z^1 - \frac{1}{2}G_2 \wedge Z^2 \end{cases}$$

The local connection 1-form  $G_\alpha, \bar{G}_\alpha$ , ( $-$  means complex conjugate) are expanded in terms of the such cobasis  $\{\theta^A\}$  by

$$(1.5) \quad G_\alpha = G_{\alpha A} \theta^A, \bar{G}_\alpha = \bar{G}_{\alpha A} \bar{\theta}^A, (\bar{\theta}^2 = \theta^3),$$

and the coefficients  $G_{\alpha A}, \bar{G}_{\alpha A}$  correspond in the (CVF) to the spinorial coefficients of Newmann-Penrose. On the other hand the forms  $\theta^A$  satisfy the following equations in terms of  $G$  and  $\bar{G}$ :

$$(1.6) \quad \begin{cases} d\theta^1 = \frac{1}{4}(G_3 + \bar{G}_3) \wedge \theta^1 + \bar{G}_1 \wedge \theta^2 + \frac{1}{2}G \wedge \theta^3 \\ d\theta^2 = -\frac{1}{2}\bar{G}_2 \wedge \theta^1 + \frac{1}{4}(G_3 - \bar{G}_3) \wedge \theta^2 + \frac{1}{2}G \wedge \theta^4 \\ d\theta^3 = -\frac{1}{2}G_2 \wedge \theta^1 - \frac{1}{4}(G_3 - \bar{G}_3) \wedge \theta^3 + \frac{1}{2}\bar{G}_1 \wedge \theta^4 \\ d\theta^4 = -\frac{1}{2}G_2 \wedge \theta^2 - \frac{1}{2}\bar{G}_2 \wedge \theta^3 - \frac{1}{2}(G_3 + \bar{G}_3) \wedge \theta^4 \end{cases}$$

The coefficients  $G_{\alpha A}, \overline{G}_{\alpha A}$ , play an important role in Petrov's classification of different types of space-times.

2. Let  $(M, g)$  be an even dimensional  $2m + 2$  pseudo-Riemannian  $C^\infty$ -manifold with Minkowski index [23], i.e. the index of the metric tensor  $g$  is  $m$  ( $m$  : number of positive squares of an orthonormal decomposition of  $g$ ). Let  $T_p(M)$  be the tangent space to  $M$  at  $p \in M$ .

By reference to [23] one has the splitting  $T_p(M) = H_p(M) \oplus S_p(M)$  where  $H_p$  and  $S_p$  are a  $2m$ -dimensional para-Hermitian vector space and a 2-dimensional spatial vector space, respectively.

Let  $\mathcal{F} = \text{vect}\{\xi_a, \xi_{a^*}, \xi_r : a = 1, \dots, m, a^* = a + m + 1, r, s = m + 1, 2m + 2\}$  be an adapted normed vector basis such that the vector fields  $\xi_\alpha, \alpha \in \{a, a^*\}$ , are *real null* vectors whilst  $\xi_r$  define an orthonormal vector basis of *space-like* vectors. Since  $\mathcal{F}$  is normed, one has [23]:

$$(2.1) \quad g(\xi_a, \xi_{b^*}) = \delta_{ab}, \quad g(\xi_\alpha, \xi_r) = 0, \quad \text{and}$$

$$(2.2) \quad g(\xi_r, \xi_s) = -\delta_{rs}.$$

We recall that  $H_p(M)$  admits the decomposition of P. Libermann [14]  $H_p = \Sigma_p \oplus \Sigma_p^*$ , where  $\Sigma_p$  and  $\Sigma_p^*$  are two *self-orthogonal* (abbr. SO)  $m$ -distributions defined by  $\{\xi_a\}$  and  $\{\xi_{a^*}\}$  respectively. The pair  $(\Sigma_p, \Sigma_p^*)$  defines an *involutive automorphism*  $J$  such that

$$(2.3) \quad J^2 = Id, \quad J\xi_a = \xi_a, \quad J\xi_{a^*} = -\xi_{a^*}.$$

The (1,1) tensor field  $J$  is called the *paracomplex operator*, and in the case under discussion, one also has  $J\xi_r = 0$ .

If  $\mathcal{F}^* = \text{covect}\{\omega^a, \omega^r\}$  denotes the associated cobasis of  $\mathcal{F}$ , then the soldering form of  $M$  (i.e. the canonical vector valued 1-form of  $M$ ), is expressed by

$$(2.4) \quad dp = \omega^a \otimes \xi_a + \omega^{a^*} \otimes \xi_{a^*} + \omega^r \otimes \xi_r.$$

Taking account of (2.1), (2.2), one has

$$(2.5) \quad g = \langle dp, dp \rangle = 2 \sum \omega^a \otimes \omega^{a^*} - \sum (\omega^r)^2, \quad \text{where}$$

$$(2.6) \quad g_h = 2 \sum \omega^a \otimes \omega^{a^*}, \quad \text{and}$$

$$(2.7) \quad g_s = -\sum (\omega^r)^2,$$

denote the para-Hermitian and the spatial component, respectively, of the metric tensor  $g$ . Next the 2-form  $\Omega$  of rank  $2m$  exchangeable with  $g_h$ , expressed by

$$(2.8) \quad \Omega = \sum \omega^a \wedge \omega^{a^*},$$

is called the *parahermitian 2-form* (abbr. pH) of  $M$ . In the following

we shall set  $\omega^r = \eta^r$  and assume that the spatial dual forms  $\eta^r$  define a 2-paracontact structure on  $M$ , that is [9],

$$(2.9) \quad d\eta^r = 0, \Omega^m \wedge \eta^{m+1} \wedge \eta^{2m+2} \neq 0, \quad r = m+1, 2m+2.$$

Matters being so, we agree also to call the spatial 2-distribution  $S_p$ , the contact distribution and write  $S_p = D_C$ . Further by reference to [9] one has

$$(2.10) \quad \nabla \xi_r = f_r(dp - l_C), \text{ where}$$

$$(2.11) \quad l_C = \eta^r \otimes \xi_r$$

is called the contact (or spatial) line element. In addition one has

$$(2.12) \quad df_p = c\eta^r, \quad c = \text{const}, \text{ and (see [9])}$$

$$(2.13) \quad \xi = \sum f_r \xi_r \in D_C,$$

$$(2.14) \quad \eta = f_r \eta^r \in D_C^*,$$

are called the bicontact vector field and the bicontact 1-form, respectively.

We also get

$$(2.15) \quad \langle \xi, \xi \rangle = -\sum f_r^2 = -2f, \quad \langle \rangle \text{ instead of } g,$$

and therefore one has

$$(2.16) \quad df = c\eta,$$

If  $X_C, X'_C \in D_C$  are any vector fields of  $D_C$ , then by virtue of (2.10) it is easily seen that one has  $\nabla_{X'_C} X_C \in D_C$ . This proves as is known that  $D_C$  is an autoparallel foliation and that the leaves  $M_C$  of  $D_C$  are totally geodesic submanifolds (surfaces) of  $M$  (see also [13]). Let now

$$(2.17) \quad \nabla \xi = \theta \otimes \xi,$$

$$(2.18) \quad d\omega = -\theta \wedge \omega, \text{ and}$$

(2.19)  $d\theta = -\theta \wedge \theta + \Theta$ , be E. Cartan's structure equations written in indexless manner, where  $\theta$  (resp.  $\Theta$ ) are the local connection forms in the bundle  $F(M)$  (resp. the curvature forms on  $M$ ). We agree to denote by  $\theta_b^{\cdot a}, \theta_b^{a \cdot}$  the mixed para-Hermitian connection forms and set  $\theta_m \in \{\theta_b^{\cdot a}, \theta_b^{a \cdot}\}$ . Further we call  $\theta_r^\alpha, \theta_\alpha^r$  and  $\theta_s^r$  the mixed spatial connection forms and the (unique) spatial torsion 1-form [25], respectively.

By (2.1), (2.2), (2.10), and making use of (2.17), one gets

$$(2.20) \quad \begin{cases} \theta_b^{\cdot a} + \theta_a^{\cdot b} = 0, \theta_b^{a \cdot} + \theta_a^{b \cdot} = 0, \theta_b^\alpha + \theta_{\alpha \cdot}^b = 0 \\ \theta_r^\alpha = -\theta_\alpha^r = f_r \omega^\alpha, \theta_s^r = 0 \end{cases}$$

If  $\Theta_{m+1}^{2m+2}$  denotes the curvature corresponding to the surface  $M_C$ , then making use of (2.19), one finds by (2.20) that  $\Theta_{m+1}^{2m+2} = 0$ . Hence we conclude that  $M_C$  is a flat surface.

By Frobenius theorem it follows that the para-Hermitian distribution  $H_p$  which annihilates the closed 2-form  $\varphi = \eta^{m+1} \wedge \eta^{2m+2}$  (the simple unit form corresponding to the contact distribution  $D_C$ ) is also involutive. Denoting by  $M_h$  the  $2m$ -dimensional leaf of  $H_p$  it follows that the space-like vectors  $\xi_r$  are normal sections on  $M_h$ . Next since the restriction  $l_C | M_h$  vanishes it follows at once from (2.10) that  $M_h$  is a *totally umbilical submanifold* of  $M$ , i.e., the second fundamental quadratic forms  $\langle dp_h, \nabla \xi_r \rangle$ , where  $dp_h = dp | M_h$ , are conformal to  $g_h$ , (see [4]).

Furthermore, since the soldering form  $dp_h$  of  $M_h$  is expressed by

$$(2.21) \quad dp_h = \omega^a \otimes \xi_a + \omega^{a^*} \otimes \xi_{a^*},$$

it follows that the mean curvature vector valued  $(2m-1)$ -form  $\mathbf{H} \in A^{2m-1}(M, TM)$  of  $M_h$  is defined by (see [2], [24])

$$(2.22) \quad \mathbf{H} = *dp_h = \sum (-1)^{a-1} \omega^1 \wedge \dots \wedge \hat{\omega}^a \wedge \dots \wedge \omega^m \wedge \omega^{m+2} \wedge \dots \wedge \omega^{2m+1} \otimes \xi_{a^*} + \sum (-1)^{a^*-1} \omega^1 \wedge \dots \wedge \omega^m \wedge \omega^{m+2} \wedge \dots \wedge \hat{\omega}^{a^*} \wedge \dots \wedge \omega^{2m+1} \otimes \xi_a.$$

In the above equation,  $*$  and the roof  $\hat{\phantom{a}}$  means the *star isomorphism* and the missing term, respectively, and in order to symplify, we have denoted the restriction on  $M_h$  by the same letters.

Operating on  $\mathbf{H}$  by  $d^\nabla$  one has by the known general formula

$$d^\nabla \mathbf{H} = 2mH \otimes \sigma_h,$$

where  $H$  (resp.  $\sigma_h$ ) means the *mean curvature vector field* (resp. the local Riemannian volume element of  $M_h$ ). Coming back to the case under discussion one finds making use of (2.18) and taking account of (2.20)

$$d^\nabla \mathbf{H} = 0 \implies H = 0,$$

which as is known shows that  $M_h$  is a *minimal* submanifold of  $M$ .

We shall now discuss some significant properties of the bicontact vector field  $\xi$  defined by (2.13). By reference to (2.10), (2.12) one derives from (2.13):

$$(2.23) \quad \nabla \xi = 2fdp + (c - 2f)l_c.$$

On the other hand denoting by  $\sigma$  the local Riemannian volume element of  $M$  one gets by (2.20) after a standard calculation:

$$(2.24) \quad \mathcal{L}_\xi \sigma = 2c\eta \implies \text{div} \xi = 2c,$$

and one may say that  $\xi$  defines an *infinitesimal homothety* on  $M$ .

Next by (2.23) one gets at once

$$(2.25) \quad \nabla_\xi \xi = c\xi$$

and making use of K. Yano's general formula adapted to the considered  $\mathcal{F}$ -basis, that is,

$$(2.26) \operatorname{div}(\nabla_{\xi}\xi) - \operatorname{div}((\operatorname{div}\xi)\xi) + (\operatorname{div}\xi)^2 = \operatorname{Ric}(\xi, \xi) + \sum \langle \nabla_{\xi_a}\xi, \xi_{a^*} \rangle \langle \nabla_{\xi_{a^*}}\xi, \xi_a \rangle + \sum \langle \nabla_{\xi_p}\xi, \xi_p \rangle^2$$

one finds taking account of (2.23), (2.24) and (2.25):

$$(2.27) \operatorname{Ric}(\xi, \xi) = -4mf^2.$$

Since  $\operatorname{Ric}$  denotes the *Ricci tensor* it follows at once from (2.15):

$$(2.28) \operatorname{Ric}(\xi) = 2mf = -m\|\xi\|^2.$$

( $\operatorname{Ric}(Z)$ ): Ricci curvature with respect to  $Z$ ,  $Z \in \mathfrak{X}M$ )

Further let  $\mu_{\varphi} : TM \rightarrow T^*M, Z \rightarrow i_Z\varphi$  be the bundle isomorphism defined by the 2-form  $\varphi = \eta^{m+1} \wedge \eta^{2m+2}$ . We agree to set

(2.29)  $\bar{\eta} = \mu_{\varphi}(\xi) = i_{\xi}\varphi = f_{m+1}\eta^{2m+2} - f_{2m+2}\eta^{m+1}$  and the dual vector field  $\bar{\xi}$  of  $\bar{\eta}$ , by

$$(2.30) \bar{\xi} = b^{-1}(\bar{\eta}) = f_{m+1}\xi_{2m+2} - f_{2m+2}\xi_{m+1}$$

Taking the covariant differential of  $\bar{\xi}$  one quickly gets by (2.10) and (2.12),

$$(2.31) \nabla\bar{\xi} = c(M^{m+1} \otimes \xi_{2m+2} - \eta^{2m+2} \otimes \xi_{m+1}) = c(\xi_{2m+2} \wedge \xi_{m+1}),$$

where  $\wedge$  means the *wedge product* (as is known  $\wedge$  is skew-symmetric with respect to  $\langle \cdot \rangle$ ).

Since from (2.31) one derives at once  $\langle \nabla_Z\bar{\xi}, Z' \rangle + \langle \nabla_{Z'}\bar{\xi}, Z \rangle = 0$ , we see that the vector field  $\bar{\xi}$  is a *Killing vector*. It also should be noticed that by (2.23) and (2.3) one finds the relation

$$\nabla_{JZ}\xi = J\nabla_Z\xi, \quad Z \in \mathfrak{X}M,$$

which is similar to that of cosymplectic quasi Sasakian manifolds [7].

**Theorem.** *Let  $(M, g)$  be a pseudo-Riemannian manifold with Minkowski index and endowed with a 2-paracontact (abb. 2-PC) spatial structure. Any such  $M$  can be viewed as the Riemannian product  $M = M_h \times M_C$  such that:*

i)  $M_C$  is a totally geodesic flat surface tangent to the spatial 2-paracontact distribution  $D_C$ .

ii)  $M_h$  is a minimal totally umbilical 2-codimensional para-Hermitian submanifold of  $M$ .

Let  $\xi$  and  $\varphi$  be the bicontact vector field defined by the (2-PC) spatial structure and the Riemannian volume element of  $M_C$  respectively. Then :

1.  $\xi$  defines an infinitesimal homothety on  $M$ , and  $\operatorname{Ric}(\xi) = -\|\xi\|^2$ ,

2. The dual vector field  $\bar{\xi}$  of  $\xi$  with respect to  $\varphi$  is a Killing vector field.

3. One may associate with the paracomplex operator  $J$  the antiinvariant operator  $\mathcal{A}$ , [26], [27] such that

$$(3.1) \mathcal{A}^2 = +1, \mathcal{A} \circ J + J \circ \mathcal{A} = 0,$$

and  $\mathcal{A}$  is symplectically anticanonic, i.e.  $\mathcal{A}\Omega = -\Omega$ .

Denote by  $\theta_m \in \{\theta_b^a, \theta_a^b\}$  the mixed connection forms associated with the Witt basis  $W = \{\xi_a, \xi_b\}$ . Consider on  $M$  the globally defined parahermitian vector field

$$(3.2) V = \sum(t_a \cdot \xi_a + t_a \xi_{a^*}), t_a, t_{a^*} \in C^\infty M, \text{ and denote by}$$

$$(3.3) v = \flat(V) = \sum t_a \omega^a$$

its dual form. It is easily checked that one has  $v(V) = 2 \sum t_a t_{a^*} = \|V\|^2$  and we agree to set

$$(3.4) t = \sum t_a t_{a^*}.$$

Recalling that the wedge product is the linear operator  $\wedge$  defined by  $(X \wedge Y)Z = g(Z, Y)X - g(X, Z)Y$ , we assume now that the mixed connection forms  $\theta_m$  are expressed by

$$(3.5) \begin{cases} \langle \mathcal{A}V, \xi_b \wedge \xi_a \rangle = \theta_b^a \\ \langle \mathcal{A}V, \xi_{b^*} \wedge \xi_{a^*} \rangle = \theta_b^{a^*} \end{cases}$$

(see also [20]). Since

$$(3.6) \mathcal{A}V = \sum(t_a \xi_a + t_{a^*} \xi_{a^*})$$

one gets by (2.15)

$$(3.7) \begin{cases} \theta_b^a = t_b \cdot \omega^a - t_a \cdot \omega^b \\ \theta_b^{a^*} = t_b \omega^{a^*} - t_a \omega^{b^*} \end{cases}$$

and one checks that equations (3.7) are matching the general relation

$$\theta_b^a + \theta_a^b = 0, \theta_b^{a^*} + \theta_a^{b^*} = 0.$$

It also should be noticed that the connection forms  $\theta_m$  are integral relations of invariance for the vector field  $V$  (in the sense of A. Lichnerowicz [15]), i.e.

$$(3.8) \theta_m(V) = 0.$$

Denote now by  $\Sigma_p$  and  $\Sigma_p^*$  the two self-orthogonal distribution [14] defined by  $\{\xi_a\}$  and  $\{\xi_a^*\}$  respectively (i.e.  $\Sigma_p = \Sigma_p^\perp$ ,  $\Sigma_p^* = \Sigma_p^{*\perp}$ ), and let

$$(3.9) \varphi = \omega^1 \wedge \dots \wedge \omega^m, \text{ and}$$

$$(3.10) \varphi^* = \omega^{m+2} \wedge \dots \wedge \omega^{2m+1}$$

be the simple unit forms corresponding to  $\Sigma_p$  and  $\Sigma_p^*$ , respectively.

Taking the exterior differential of  $\Sigma_p$  and  $\Sigma_p^*$ , one derives by (2.14), (3.3), (3.7) and making use of equations (2.16)

$$(3.11) \begin{cases} d\varphi = (v + \eta - \theta_{Ric}) \wedge \varphi \\ d\varphi^* = (v + \eta + \theta_{Ric}) \wedge \varphi^* \end{cases}$$

which shows that both  $m$ -forms  $\varphi, \varphi^*$  are *exterior recurrent* [6]. In the above equations  $\theta_{Ric} = \sum \theta_a^a, (\theta_a^a = -\theta_a^{a*})$ , denotes the *Ricci 1-form* of  $M$ , [2] and by Frobenius theorem one may say that  $\Sigma_p$  and  $\Sigma_p^*$  define each a *self-orthogonal (abbr. SO) foliation*. Therefore we agree to denominate the connection defined by (3.5) a *(SO)-foliate connection* (see also [20]). Due to the fact that  $\eta$  is a closed form, it follows by a simple argument from (3.11), that both 1-forms  $v$  and  $\theta_{Ric}$  are closed.

On the other hand taking the exterior differential of the para-Hermitian 2-form  $\Omega$  defined by (2.8), one deduces taking account of (3.7)

$$(3.12) \quad d\Omega = 2(v + \eta) \wedge \Omega.$$

Therefore since  $\Omega$  is of *rank*  $= 2m = \dim M - 2$ , we agree to call  $\Omega$  a *conformal 2-cosymplectic form* [9]. In consequence of this fact the 1-form  $v + \eta$ , may be considered as the *Lee covector* associated with  $\Omega$ . In the light of this definition and by reference to the concept of *f-Kenmotsu manifolds* [18], [16], one may define  $M$  as a *2-paracontact f-Kenmotsu pseudo-manifold*.

Consider now at each point  $p$  of  $M$  the two complementary distributions:

$$\mathcal{D}_p = \text{vect}\{\xi_a, \xi_{m+1}, a = 1, \dots, m\},$$

$$\mathcal{D}_p^* = \text{vect}\{\xi_{a^*}, \xi_{2m+2}, a^* = m+2, 2m+1\},$$

$$\text{and denote by } \sigma_C = \varphi \wedge \eta^{m+1}, \sigma_{C^*} = \varphi^* \wedge \eta^{2m+2}$$

the simple unit forms corresponding to  $\mathcal{D}_p$  and  $\mathcal{D}_p^*$ , respectively.

By (2.10) and (3.11) one quickly gets

$$(3.13) \begin{cases} d\sigma_C = (v + \eta - \theta_{Ric}) \wedge \sigma_C \\ d\sigma_{C^*} = (v + \eta + \theta_{Ric}) \wedge \sigma_{C^*} \end{cases}$$

which show that both  $(m+1)$ -forms  $\sigma_C$  and  $\sigma_{C^*}$  are exterior recurrent. Since  $\sigma_C$  (respectively  $\sigma_{C^*}$ ) annihilates  $\mathcal{D}_p^*$  (respectively  $\mathcal{D}_p$ ), it follows by Frobenius theorem that both  $\mathcal{D}_p$  and  $\mathcal{D}_p^*$  define  $(m+1)$ -foliations.

Consider for instance the foliation  $\mathcal{D}_p$ , and denote by  $\mathcal{D}_p^\perp$  the orthogonal distribution to  $\mathcal{D}_p$ . Clearly by (2.1) one has  $\mathcal{D}_p^\perp \subset \mathcal{D}_p$ , which shows that  $\mathcal{D}_p$  is a *coisotropic foliation* [24]. Obviously  $\mathcal{D}_p^*$  enjoys the same property. Denote

by  $M_I$  and  $M_I^*$  the  $(m+1)$ -dimensional leaves of  $\mathcal{D}_p$  and  $\mathcal{D}_p^*$  respectively, and by

$$dp_I = \omega^a \otimes \xi_a + \eta^{m+1} \otimes \xi_{m+1}$$

and

$$dp_I^* = \omega^{a^*} \otimes \xi_{a^*} + \eta^{2m+2} \otimes \xi_{2m+2}$$

the corresponding soldering forms.

Since  $\xi_r$  ( $r = m+1, 2m+2$ ) are the only *anisotropic vectors* of  $dp_I$  and  $dp_I^*$ , it follows that the metric tensors of  $M_I$  and  $M_I^*$  are expressed by  $g_I = (\eta^{m+1})^2$ ,  $g_I^* = (\eta^{2m+2})^2$  (we denote the induced element on  $M_I$  and  $M_I^*$  by the same letters). Therefore  $M$  may also be viewed as the Riemannian product  $M = M_I \times M_I^*$  where  $M_I$  and  $M_I^*$  are coisotropic and of defect  $d = m$ , submanifolds of  $M$  ( $d = \dim M_I - \text{rank } g_I$ , etc.)

It should be noticed that  $M_I$  and  $M_I^*$  can be also regarded as *antiinvariant submanifolds* of  $M$  [28]. Effectively, let us consider  $M_I$  and denote by  $T_{p_I}(M_I)$  and  $T_{p_I}^\perp(M_I)$  the tangent space and the normal space respectively at any point  $p_I \in M_I$ .

Then since by (2.3) one has  $J\xi_a = \xi_a$ ,  $J\xi_{a^*} = -\xi_{a^*}$ , it follows  $JT_{p_I}(M_I) = T_{p_I}^\perp(M_I)$ , which proves the above assertion (see also [9]).

In view of the next discussion it is convenient to expand the structure equations (2.17), (2.18) corresponding to the Witt vector and covector basis as follows

$$(3.14) \begin{cases} \nabla \xi_a = \theta_a^b \otimes \xi_b - \omega^a \otimes \xi + t_a dp_{\Sigma^*} - \omega^{a^*} \otimes V_{\Sigma^*} \\ \nabla \xi_{a^*} = \theta_{a^*}^b \otimes \xi_b - \omega^{a^*} \otimes \xi + t_{a^*} dp_{\Sigma} - \omega^a \otimes V_{\Sigma} \end{cases}$$

and

$$(3.15) \begin{cases} d\omega^a = \omega^b \wedge \theta_b^a - \omega^a \wedge (\eta + v_{\Sigma^*}) + t_a \Omega \\ d\omega^{a^*} = \omega^{b^*} \wedge \theta_{b^*}^{a^*} - \omega^{a^*} \wedge (\eta + v_{\Sigma}) - t_a \Omega \end{cases}$$

where we have set

$$(3.16) \quad dp_{\Sigma} = \omega^a \otimes \xi_a, dp_{\Sigma^*} = \omega^{a^*} \otimes \xi_{a^*}$$

$$(3.17) \quad V_{\Sigma} = \Sigma t_a \cdot \xi_a, V_{\Sigma^*} = t_a \xi_{a^*}, \text{ and}$$

$$(3.18) \quad v_{\Sigma} = \Sigma t_a \omega^a, v_{\Sigma^*} = \Sigma t_{a^*} \omega^{a^*}$$

On the other hand by (3.12) we have seen that the dual form  $v$  of  $V$  is closed ( $dv = 0$ ).

We assume in this paper that  $v$  and  $-2\eta$  are not *homologous*. In this conditions one derives from (3.3) and making use of (3.15),

$$(3.19) \begin{cases} dt_a = t_b \theta_a^b - t_a \eta \\ dt_{a^*} = t_b \theta_{a^*}^b - t_{a^*} \eta. \end{cases}$$

Next by (3.19) and (3.4) one quickly gets

$$(3.20) \quad dt + 2t\eta = 0$$

If  $\mu_\Omega : Z \rightarrow i_Z \Omega$  denotes the bundle isomorphism defined by  $\Omega$ , one has  $\mu_\Omega Z = b^{-1}(JZ) = \mu Z(JZ) = -g(Z, Z)$ ,

and in the case under discussion we agree to set

$$(3.21) \quad \mu V = \bar{v} = \sum t_{a^*} \omega^{a^*} - t_a \omega^a$$

Further taking the exterior differential of  $\bar{v}$ , one derives by (3.15) and (3.19):

$$(3.22) \quad d\bar{v} = -2t\Omega + v \wedge \bar{v}$$

From the above equations and (3.12) it follows at once, by  $d(d\bar{v}) = 0$

$$(3.23) \quad t = 0 \implies \|V\|^2 = 0,$$

and the above reveals that  $V$ , which we agree to denominate the parahermitian Lee vector, is a null (real) vector field.

In consequence of (3.23), one may write

$$(3.24) \quad d^{-v}\bar{v} = 0$$

which shows that  $\bar{v}$  is a  $d^{-v}$ -closed form (see (1.1)). In this conditions one quickly finds by (3.12) that the Lie derivative  $\mathcal{L}_V \Omega$  is expressed by  $\mathcal{L}_V \Omega = -(2\eta - v) \wedge \bar{v}$

and performing the exterior differentiation of (3.25) we get after a short calculation:

$$(3.26) \quad d(\mathcal{L}_V \Omega) = v \wedge \mathcal{L}_V \Omega \iff d^{-v}(\mathcal{L}_V \Omega) = 0$$

Hence the Lie derivative  $\mathcal{L}_V \Omega$  enjoys as  $\mu V$  the property be  $d^{-v}$ -closed (see also [2],[17]). One may also define  $V$  as infinitesimal relative conformal transformation [22] of  $\Omega$ .

Further, let  $L$  be the (1,1)-operator defined [8] by  $L : u \rightarrow u \wedge \Omega$ , (note that one has  $dLu = Ldu + v \wedge Lu$ ) and set

$$(3.27) \quad \bar{v}_q = \bar{v} \wedge \Omega^q, (\bar{v}_1 = L\bar{v})$$

One derives

$$(3.28) \quad d\bar{v}_q = ((2q+1)v + 2q\eta) \wedge \bar{v}_q \iff d^{-((2q+1)v+2q\eta)}\bar{v}_q$$

which shows that all the  $(2q+1)$ -forms  $\bar{v}_q$  are  $d^{-((2q+1)v+2q\eta)}$ -closed.

Next, since  $\bar{v}(V) = 0$ , one gets at once by (3.24),  $\mathcal{L}_V \bar{v} = 0$ , and by a standard calculation, that

$$(3.29) \quad \mathcal{L}_V \bar{v}_q = 0.$$

Hence  $V$  defines an *infinitesimal automorphism* of all the  $(2q+1)$ -forms  $\bar{v}_q$ . On the other hand, since  $\mu(JV) = i_{JV}V = v, v(JV) = 0$ , one gets instantly, by (3.12),

$$(3.30) \quad \mathcal{L}_{JV}\Omega = 0$$

that is  $JV$  defines an infinitesimal automorphism of  $\Omega$ . Noticing that  $\|v\|^2 \implies \|JV\|^2 = 0$ , it is a simple matter to show that one also has  $\mathcal{L}_{JV}\bar{v}_q = 0$ .

Let now  $X \in D_C$  be any space-like vector field. Then, since  $i_X\Omega = 0, i_Xv = 0$ , one gets instantly  $\mathcal{L}_X\Omega = g(X, \xi)\Omega$ , i.e. any  $X$  defines an *infinitesimal conformal transformation* of  $\Omega$ .

In the light of the same problem, one finds by (3.15) and (3.19)

$$(3.31) \quad \begin{cases} d\theta_a^{b*} = (\omega_\Sigma + \theta_{Ric}) \wedge \theta_a^{b*} \\ d\theta_a^b = (\omega_{\Sigma^*} - \theta_{Ric}) \wedge \theta_a^b \end{cases}$$

which shows that all the mixed connection forms  $\theta_m$  are exterior recurrent.

Since we have seen that  $\theta_m(V) = 0$ , and clearly  $\omega_\Sigma(V) = \omega_{\Sigma^*}(V) = t = g(V, V)Z = 0$ , one derives from (3.31) by a short calculation,

$$(3.32) \quad \begin{cases} \mathcal{L}_V\theta_a^{b*} = \theta_{Ric}(V)\theta_a^{b*} \\ \mathcal{L}_V\theta_a^b = -\theta_{Ric}(V)\theta_a^b \end{cases}$$

Therefore one may say that the vector field  $V$  defines an *infinitesimal conformal transformation* of all the mixed connection forms  $\theta_m$ , with  $\pm\theta_{Ric}(V)$  as conformal scalar (by a suitable choice of indices).

Next remembering that  $\|V\|^2 = 0 = \|JV\|^2$ , one derives by (3.14) and (3.19) that

$$(3.37) \quad \begin{cases} \nabla V = -(\eta + \frac{\nu}{2}) \otimes V + \frac{\bar{\nu}}{2} \otimes JV + \alpha \otimes \xi \\ \nabla JV = -(\eta - \frac{\nu}{2}) \otimes JV - \frac{\bar{\nu}}{2} \otimes V + \bar{\alpha} \otimes \xi \end{cases}$$

where we have set

$$(3.38) \quad \begin{cases} \alpha = \mu(\mathcal{A}JV) = -\sum(t_{a^*}\omega^a + t_a\omega^{a^*}) \\ \bar{\alpha} = \mu(\mathcal{A}V) = \sum(t_a\omega^{a^*} - t_{a^*}\omega^a) \end{cases}$$

In addition, an easy calculation gives

$$(3.39) \begin{cases} \alpha(JV) = \bar{\alpha}(V) = \sum(t_a)^2 - \sum(t_{a^*})^2 \\ \alpha(V) = \bar{\alpha}(JV) = -\sum(t_a)^2 + \sum(t_{a^*})^2 \end{cases}$$

and by (3.37) and (3.38) we find  $[V, JV] = 0$ , that is  $V$  and  $JV$  commute. By (2.21) and (3.37) we check the general formula (Ricci identity)

$$(\mathcal{L}_X g)(U, U') = \langle \nabla_U X, U' \rangle + \langle U, \nabla_{U'} X \rangle, \quad X, U, U' \in \mathfrak{X}M$$

by setting  $X = \xi$ ,  $U = V$ ,  $U' = JV$ . Now, making use of the general formula  $\operatorname{div} Z = \operatorname{tr} \nabla Z$ , i.e. in the case under discussion

$$\operatorname{div} V = \sum \langle \nabla_{\xi_a} V, \xi_{a^*} \rangle + \sum \langle \nabla_{\xi_{a^*}} V, \xi_a \rangle + \sum \langle \nabla_{\xi_r} V, \xi_r \rangle$$

one finds by reference to (3.37) that  $\operatorname{div} V = 0$  and in similar way  $\operatorname{div} JV = 0$ .

Hence  $V$  and  $JV$  define both infinitesimal automorphisms of the volume element of  $M$ .

**Theorem.** *Let  $(M, g)$  be a  $(2m + 2)$ -dimensional pseudo-Riemannian manifold with Minkowski index. If  $M$  carries a 2-paracontact spatial structure  $D_C$  and is endowed with a self-orthogonal connection then  $M$  enjoys the following properties:*

i)  $M$  may be viewed as the Riemannian product  $M = M_I \times M_I^*$  where  $M_I$  and  $M_I^*$  are  $(m + 1)$ -dimensional coisotropic submanifolds of defect  $d = m$ .

ii)  $M$  is endowed with a local conformal 2-cosymplectic structure defined by the parahermitian 2-form  $\Omega$ .

iii) The para-Hermitian component  $V$  of the Lee vector field, associated with this structure is a null vector field, and  $V$  and  $JV$  commute ( $J$ : para-complex operator) and they define the following infinitesimal transformation:

1)  $V$  defines an infinitesimal relative conformal transformation of  $\Omega$ , i.e.  $d(\mathcal{L}_V \Omega) = v \wedge \mathcal{L}_V \Omega$ ,  $v = b(V)$ ;

2)  $V$  defines an infinitesimal automorphism of all the  $(2q + 1)$ -forms  $\bar{v}_q$ , i.e.  $\mathcal{L}_V \bar{v}_q = 0$  where  $\bar{v}_q = \bar{v} \wedge \Omega^q$ ,  $\bar{v} = \mu V = i_V \Omega$ ;

3)  $V$  defines an infinitesimal conformal transformation of all the mixed connection forms  $\theta_m$  with  $\pm \theta_{\operatorname{Ric}}(V)$  as conformal scalar, where  $\theta_{\operatorname{Ric}}$  is the Ricci 1-form;

4)  $JV$  defines an infinitesimal automorphism of  $\Omega$ , i.e.  $\mathcal{L}_{JV} \Omega = 0$ .

4. It has been outlined in [23] that a 4-dimensional pseudo-Riemannian manifold  $M$  with Minkowski index is a general space-time manifold. In the

following discussion we shall make use of the complex vectorial formalism (abbr. CVF) developed in [5] (see Preliminaries).

Let now  $(M, g)$  be a general space-time satisfying the usual integrability conditions and consider on  $M$  a vectorial basis as in [5], denoted by  $S = \text{vect}\{h_A, A = 1, \dots, 4, \}$  and let  $S^* = \text{covect}\{\theta^A\}$  be its associated cobasis.

We recall that  $h_1, h_4$  are *real null* vector fields, whilst  $h_2, h_3$  are *complex conjugate null* vector fields. They satisfy the following normalization conditions.

$$(4.1) \quad \langle h_1, h_4 \rangle = 1, \langle h_2, h_3 \rangle = -1, (\langle, \rangle \text{ instead of } g)$$

Accordingly, the soldering form  $dp$  of  $M$  is expressed in terms of the  $S$ -basis by

$$(4.2) \quad dp = \theta^A \otimes h_A$$

and as a consequence of (4.1) the metric tensor  $g$  is expressed by

$$(4.3) \quad g = \langle dp, dp \rangle = 2(\theta^1 \otimes \theta^4 - \theta^2 \otimes \theta^3)$$

Let us now come back to the case under discussion, that is  $m = 1$ . By reference to (2.1) and (2.2) the change of frame  $\mathcal{F} \rightarrow S$  is defined by

$$(4.4) \quad \xi_1 = h_1, \xi_3 = h_4, \xi_2 = \frac{i(h_2 - h_3)}{\sqrt{2}}, \xi_4 = \frac{h_2 + h_3}{\sqrt{2}}.$$

Since  $\langle \xi_2, \xi_2 \rangle = -1 = \langle \xi_4, \xi_4 \rangle$  it is seen by (4.4) that the vectors  $h_2, h_3$  determine the spatial (or paracontact) distribution  $D_C$  defined in Section 2. Next by (2.4), (4.2) and (4.4), it is an easy matter to see that the transformation  $\mathcal{F}^* \rightarrow \varphi^*$  is expressed by

$$(4.5) \quad \begin{cases} \omega^1 = \theta^1, \omega^3 = \theta^4 \\ \omega^2 = \frac{1}{\sqrt{2}}(\theta^3 - \theta^2), \omega^4 = \frac{1}{\sqrt{2}}(\theta^3 + \theta^2) \end{cases}$$

and by reference to the notation from Section 2, one may set  $\omega^2 = \eta^2, \omega^4 = \eta^4$ . Therefore by (2.12) and (4.5) it follows at once

$$(4.6) \quad d\theta^2 = 0, d\theta^3 = 0$$

Now from of equations 1, it follows from (4.6) that the (CVF) connection forms  $G_\alpha, \bar{G}_\alpha, (\alpha = 1, 2, 3)$  satisfy in the case under discussion the equations

$$(4.7) \quad G_1 = 0, G_2 = 0, \bar{G}_1 = 0, \bar{G}_2 = 0, G_3 = \bar{G}_3.$$

By reference to [12], equations (4.7) prove the following significant fact: If the pseudo-Riemannian manifold with Minkowski index defined in Section 2, has dimension 4, then, it is a *conformal recurrent* space-time.

The following fact should also be outlined. Since the component  $Z^3$  of the dual basis of  $\mathbb{C}$  is expressed (see (1.3)) by  $Z^3 = \frac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3)$ , then clearly

$Z^3$  may be considered as a local almost symplectic form (it is exchangeable with  $g$ ) on the space-time  $M$ .

But in case equations (4.7) are satisfied, it is easily seen by (1.6) that  $dZ^3 = 0$ . Therefore by reference to [27] one may say that the considered space-time is endowed with a local symplectic structure and is of Schwarzschild type.

**Theorem.** *Any 4-dimensional pseudo-Riemannian manifold with Minkowski index endowed with a spatial 2-paracontact structure, is a conformal recurrent space-time.*

## References

- [1] Buchner K., Goldberg V.V. and Rosca R., *Biconformal cosymplectic manifolds*, Riv. Mat. Univ. Parma 4 (1991), 41-58.
- [2] Buchner K. and Rosca R., *Co isotropic submanifolds of a para Kählerian manifold with concircular structure vector field*, J. Geom., 25 (1985) 164-177.
- [3] Bucki A., *Submanifolds of almost  $r$ -paracontact manifolds*, Tensor N.S. 40 (1984), 57-59.
- [4] Chen B. Y., *Geometry of Submanifolds*, M. Dekker, Inc., New York, 1973.
- [5] Cohen M., Debever R. and Defoisse L., *A complex vectorial formalism in general Relativity*, J. Math. Mech. 16 (1967).
- [6] Datta D. K., *Exterior recurrent forms on a manifold*, Tensor N.S 36 (1982), 115-120.
- [7] Endo H., *Invariant submanifolds in conformal flat almost cosymplectic manifold*, Tensor N.S. 46 (1986), 223-229.
- [8] Goldberg S., *Curvature and Homology*, Academic Press, New York, 1962.

- [9] Goldberg V. and Rosca R., *Almost conformal 2 - cosymplectic pseudo-Sasakian Manifolds*, Note di Matematica, **8** (1988), 123-140.
- [10] Goldberg V. V. and Rosca R., *Foliate conformal Kählerian Manifolds*, Rend. Sem. Mat. Messina ( to appear).
- [11] Guedira F. and Lichnerowicz A., *Géométrie des algèbres de Lie locales de Kirillov*, J. Math. Pures Appl. **63** (1984), 407-484.
- [12] Israel W., *Differential Forms in General Relativity*, Dublin, Institute for Advanced Studies, 1960.
- [13] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry*, Vol. II, Intersc. Publ., New York 1969.
- [14] Libermann P., *Sur le problème d'équivalence de certaines structures infinitésimales*, Ann. di Mat., **36** (1954), 24-120.
- [15] Lichnerowicz A., *Les relations intégrales d'invariance et leurs applications à la dynamique*, Bull. Sc. Math. **70** (1946), 82-95.
- [16] Matsumoto K., Mihai I. and Rosca R., *A certain locally conformal almost cosymplectic manifold and its submanifolds*, Tensor N.S. **51** (1992), 90-102.
- [17] Naitza D. and Rosca R., *On conformal symplectic Riemannian manifolds*, Atti Accad. Pel. Peric., Cl. I. Sc., Messina **66** (1988), 217-229.
- [18] Olszak Z. and Rosca R., *Normal locally conformal almost cosymplectic manifolds*, Pub. Math. Debrecen, **39**, (1991).
- [19] Poor W.A., *Differential Geometric Structures*, Mac Graw-Hill, New York, (1981).
- [20] Reyes E. and Rosca R., *Self-orthogonal foliate conformal symplectic almost para-Hermitian manifold*, to appear in Rend. Circ. Mat. Palermo.
- [21] Rosca R., *Codimension 2, CR-submanifolds with null covariant decomposable vertical distribution of a neutral manifold M*, Rend. Mat. **4** (1982), 787-797.

- [22] Rosca R., *On some infinitesimal relative conformal transformations* (preprint).
- [23] Rosca R., *Pseudo-euclidean space with Minkowski index and having the quasi-concurrence property*, Tensor N.S., **40** (1983), 21-26.
- [24] Rosca R., *Variétés neutres  $M$  admettant une structure conforme symplectique et feuilletage coisotrope*, C. R. Acad. Sci. Paris, **300** (1985), 631-634.
- [25] Rosca R. and Vanhecke L., *Les sous-variétés isotropes et pseudo-isotropes d'une variété hyperbolique à  $n$ -dimensions*, Verhand. Konink. Acad. Wetensch. België, Kl. Wetenschappen, **38** (1976), 136.
- [26] Rosca R. and Vanhecke L., *Sur une variété presque paracokählerienne munie d'une connexion self-orthogonale involutive*, Ann. St. Univ. Al. I. Cuza, Iasi, **22** (1976), 1-10.
- [27] Vranceanu G. and Rosca R., *Introduction in Relativity and Pseudo-Riemannian Geometry*, Edit. Rep. Soc. Roum., Bucarest, 1976.
- [28] Yano K. and Kon M., *CR Submanifolds of Kählerian and Sasakian Manifolds*, Birkhäuser, Progress in Mathematics, **30**, Boston, Stuttgart, 1983.