LIBERTAS MATHEMATICA, VOL. XV (1995)

SOME REMARKS ON THE PHASE FIELD EQUATIONS

Gheorghe Moroşanu

1. Introduction

In this note we are concerned with the following system of *phase field equations* [2-5]

$$
\begin{aligned}
&u_t + \frac{l}{2}\phi_t - K\Delta u = 0 \ , \\
&\tau\phi_t - \xi^2\Delta\phi - \frac{1}{2a}(\phi - \phi^3) - 2u = 0 \ , \ x \in \Omega, \ t > 0 \ ,
\end{aligned}
\tag{1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \leq 3)$ with smooth boundary Γ. This system describes the phase separation events involving the liquid-solid transitions. The unknowns u and ϕ represent the reduced temperature and phase function, respectively, while l, K, τ, ξ, a are positive constants (representing the latent heat, the thermal diffusivity, the relaxation time, a length scale and a small parameter, respectively).

Global existence and uniqueness of solution to system (1) with Dirichlet boundary conditions

$$\phi(t,x) = \phi_\Gamma(x) \ , \ \ u(t,x) = u_\Gamma(x) \ , \ \ x \in \Gamma \ , \ \ t > 0 \ , \tag{2}$$

and initial conditions

$$\phi(0,x) = \phi_0(x) \ , \ \ u(0,x) = u_0(x) \ , \ \ x \in \Omega \ , \tag{3}$$

have been proven by G. Caginalp [3], C.M. Elliott and Zheng Songmu [5], P.W. Bates and Zheng Songmu [2].

In this note we derive the same result by using the theory of evolution equations associated to ω-monotone operators. One of the advantages of this method is that the fractional step scheme as developed by V. Barbu and M. Iannelli [1] can be applied directly to approximate the solution of (1), (2), (3).

2. Existence and uniqueness of solution

The main result of this section is

Theorem 2.1 *For any $u_0, \phi_0 \in H^2(\Omega)$ and $u_\Gamma, \phi_\Gamma \in H^{3/2}(\Gamma)$ satisfying the compatibility conditions, problem (1), (2), (3) admits a unique global smooth solution .*

Proof. As remarked before, this theorem is known, but the method used here is different. Let $\overline{u}=\overline{u}(x)$, $\overline{\phi}=\overline{\phi}(x)$ be harmonic functions in Ω satisfying on the boundary Γ

$$\overline{u}\,|_{\Gamma}=u_{\Gamma}\,,\quad \overline{\phi}\,|_{\Gamma}=\phi_{\Gamma}\,.$$

Denoting $\psi=\phi-\overline{\phi}$ and $y=u-\overline{u}+(l/2)\psi$ we can see that problem (1), (2), (3) becomes

$$\begin{aligned} &y_t-K\Delta y+(Kl/2)\Delta\psi=0\\ &\psi_t-(\xi^2/\tau)\Delta\psi+\tau^{-1}(l-1/2a)\psi-2\tau^{-1}y+(1/2a\tau)(\psi+\overline{\phi})^3\\ &\qquad\qquad=(1/2a\tau)\overline{\phi}+2\tau^{-1}\overline{u}\,,\qquad x\in\Omega,\ t>0\,; \end{aligned} \tag{4}$$

$$y(t,x)=0,\ \psi(t,x)=0,\ x\in\Gamma,\ t>0\,, \tag{5}$$

$$y(0,x)=y_0(x),\ \psi(0,x)=\psi_0(x),\ x\in\Omega\,, \tag{6}$$

where

$$\psi_0(x)=\phi_0(x)-\overline{\phi}(x),\ y_0(x)=u_0(x)-\overline{u}(x)+(l/2)\psi_0(x)\,.$$

Consider the space $H=L^2(\Omega)\times L^2(\Omega)$ of the pairs (y,ψ) endowed with the scalar product

$$\left\langle \binom{y}{\psi},\binom{\widetilde{y}}{\widetilde{\psi}}\right\rangle=\delta\int_\Omega y\widetilde{y}\,dx+\int_\Omega \psi\widetilde{\psi}\,dx$$

and with the corresponding Hilbertian norm, where δ is a small positive constant.

Let $A\colon D(A)\subset H\to H$ be the linear operator defined by

$$D(A)=(H^2(\Omega)\cap H_0^1(\Omega))^2\,, \tag{7}$$

$$A\binom{y}{\psi}=\begin{pmatrix} -K\Delta y+(Kl/2)\Delta\psi\\ -(\xi^2/\tau)\Delta\psi+\tau^{-1}(l-1/2a)\psi-(2/\tau)y\end{pmatrix} \tag{8}$$

We continue the proof of Theorem 2.1 with the following

Lemma 2.1 *There exists a positive constant ω such that $A+\omega I$ is a monotone (positive) operator in H, where I is the identity operator.*

Proof of Lemma 2.1 Using Green's formula we can see that for $(y,\psi)\in D(A)$ we have

$$\left\langle A\binom{y}{\psi},\binom{y}{\psi}\right\rangle=K\delta\int_\Omega \|\nabla y\|^2_{\mathbf{R}^n}dx-(K\delta l/2)\int_\Omega\nabla\psi\cdot\nabla y\,dx+(\xi^2/\tau)\int_\Omega\|\nabla\psi\|^2_{\mathbf{R}^n}dx$$
$$+\tau^{-1}(l-1/2a)\int_\Omega\psi^2dx-(2/\tau)\int_\Omega y\psi\,dx\,.$$

Therefore

$$\left\langle A\binom{y}{\psi},\binom{y}{\psi}\right\rangle\geq-(1/\tau)\int_\Omega y^2-(1/\tau)(1-l-1/2a)\int_\Omega\psi^2+K\delta\int_\Omega\|\nabla y\|^2_{\mathbf{R}^n}dx$$

$$-(K\delta l/2)\Big(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^n}dx\Big)^{1/2}\cdot\Big(\int_\Omega \|\nabla\psi\|^2_{\mathbb{R}^n}dx\Big)^{1/2}$$
$$+(\xi^2/\tau)\int_\Omega \|\nabla\psi\|^2_{\mathbb{R}^n}dx\ .$$

Hence for $\delta > 0$ small enough and for $\omega = (1/\tau)max(1, 1-l-1/2a)$, the operator $A+\omega I$ is monotone (positive). Q.E.D.

Now let $F: D(F) \subset H \to H$ be defined by

$$D(F) = L^2(\Omega)\times L^6(\Omega)\ , \tag{9}$$

$$F\binom{y}{\psi} = \begin{pmatrix} 0 \\ (1/2a\tau)(\psi+\overline{\phi})^3 \end{pmatrix}. \tag{10}$$

Obviously F is maximal monotone in H (see, e.g., [6, Chapter I]). Moreover, we have

Lemma 2.2 *Operator $A+F+\omega I$ is maximal monotone in H.*

Proof of Lemma 2.2 By the Sobolev imbedding theorem $H^1 \to L^6(\Omega)(n \le 3)$, and hence $D(A+F) = D(A)$. In order to prove maximality of $A+F+\omega I$ it suffices to show that for every $(f,g) \in H$ and for $\alpha > 0$ large enough the following boundary value problem

$$\begin{aligned} &\alpha y - \Delta y + (l/2)\Delta\psi = f\ , \\ &\alpha\psi - \Delta\psi - (2/\xi^2)y + (1/2a\xi^2)(\psi+\overline{\phi})^3 = g\ , \quad x\in\Omega\ , \end{aligned} \tag{11}$$

$$y = 0,\ \psi = 0 \text{ on } \Gamma\ , \tag{12}$$

has a solution. Using the substitution $z = y - (l/2)\psi$ we can see that (11), (12) can be equivalently written as

$$\begin{aligned} &\alpha z - \Delta z + (l\alpha/2)\psi = f\ , \\ &(\alpha - l/\xi^2)\psi - \Delta\psi - (2/\xi^2)z + (1/2a\xi^2)(\psi+\overline{\phi})^3 = g \quad \text{in } \Omega\ . \end{aligned} \tag{13}$$

$$z = 0,\ \psi = 0 \text{ on } \Gamma\ . \tag{14}$$

From the first equation of (13) we have

$$z = P\psi := (I - \alpha^{-1}\Delta)^{-1}(\alpha^{-1}f - (l/2)\psi)\ .$$

Obviously P is a Lipschitz operator in H with Lipschitz constant $l/2$. Using this formula in the second equation of (13) we obtain

$$\gamma\psi - \Delta\psi - (2/\xi^2)P\psi + (1/2a\xi^2)(\psi+\overline{\phi})^3 = g\ , \tag{15}$$

where $\gamma = \alpha - l/\xi^2$ is positive. Denote $\beta(r) = (1/2a\xi^2)r^3$. Recall that the operator

$-\Delta+\overline{\beta}$ is maximal monotone in $L^2(\Omega)$, where $\overline{\beta}$ is the canonical extension of β to the space $L^2(\Omega)$ (see, e.g., [6, pp.181-182]). Therefore the operator $\psi \rightarrow Q(\psi) := -\Delta\psi+\overline{\beta}(\psi+\overline{\phi})$ is also maximal monotone in $L^2(\Omega)$. Now, we can see that equation (15) can be written in the form

$$\psi=(I+\gamma^{-1}Q)^{-1}(\gamma^{-1}g+(2/\xi^2\gamma)P\psi)\,. \tag{16}$$

Denoting by $T\psi$ the right hand side of equation (16) we can see that for α sufficiently large (hence γ sufficiently large) T is a contraction on H and so T has a unique fixed point. Consequently equation (15) has a unique solution $\psi\in H^2(\Omega)\cap H_0^1(\Omega)$. So, the proof is complete. Q.E.D.

Proof of Theorem 2.1 (continued). Problem (4), (5), (6) can be written as a Cauchy problem in the space $H=L^2(\Omega)\times L^2(\Omega)$ for an evolution equation associated with operator $A+F$:

$$\frac{d}{dt}\binom{y}{\psi}+(A+F)\binom{y}{\psi}=\begin{pmatrix}0\\(1/2a\tau)\overline{\phi}+(2/\tau)\overline{u}\end{pmatrix},\ t>0\,, \tag{17}$$

$$\binom{y}{\psi}(0)=\begin{pmatrix}y_0\\ \psi_0\end{pmatrix}. \tag{18}$$

Taking into account Lemma 2.2 we can derive the existence and uniqueness for problem (17), (18) by using the general theory for evolution equations associated with ω-monotone operators. The smoothness of solution follows by standard arguments. Q.E.D.

Remark 2.1 It is obvious from the proof of Lemma 2.2 that $A+\omega I$ is also a maximal monotone operator.

3. Approximation by fractional step method

In this section we discuss the possibility to approximate by the fractional step method the solution (y,ψ) of problem (4), (5), (6) for y_0, ψ_0 fixed in $H^2(\Omega)\cap H_0^1(\Omega)$. We intend to apply Corollary 1 in [1] with $C=H$. To this purpose we first prove

Proposition 3.1 *For $y_0,\psi_0\in H^2(\Omega)\cap H_0^1(\Omega)$ and $T>0$ fixed the corresponding solution (y,ψ) of problem (4), (5), (6) satisfies:*

$$sup\{\,\|y(t,\cdot)\|_{L^\infty(\Omega)};\ 0\le t\le T\}<\infty\,,$$

$$sup\{\,\|\psi(t,\cdot)\|_{L^\infty(\Omega)};\ 0\le t\le T\}<\infty\,.$$

Proof. Denoting $v=y-(l/2)\psi$ we can write (4), (5), (6) in the form

$$v_t + (l/2)\psi_t - K\Delta v = 0 \text{ in } (0,T)\times\Omega\,, \tag{19}$$

$$\tau\psi_t - \xi^2\Delta\psi - (1/2a)\psi - 2v + (1/2a)(\psi+\overline{\phi})^3 = \overline{g} \text{ in } (0,T)\times\Omega\,, \tag{20}$$

$$v = 0,\ \psi = 0 \text{ in } (0,T)\times\Gamma\,, \tag{21}$$

$$v(0,\cdot) = y_0 - (l/2)\psi_0,\ \psi(0,\cdot) = \psi_0\,, \tag{22}$$

where $\overline{g} = (1/2a)\overline{\phi} + 2\overline{u}$. Following an idea from [5] we multiply equation (19) by $(4/l)v$ and equation (20) by ψ_t and then add together and integrate with respect to x:

$$\frac{d}{dx}V(t) + \int_\Omega \Big(\tau\psi_t^2 + \frac{4K}{l}\|\nabla v\|^2_{\mathbf{R}^n}\Big)dx = 0\,,$$

where

$$V(t) = \int_\Omega \Big\{\frac{\xi^2}{2}\|\nabla\psi\|^2_{\mathbf{R}^n} - \frac{1}{4a}\psi^2 + \frac{1}{8a}(\psi+\overline{\phi})^4 + \frac{2}{l}v^2 - \overline{g}\psi\Big\}dx\,.$$

Using this fact and a reasoning similar to that in [5] we can obtain the conclusion. Q.E.D.

Denoting $M := sup\{\|\phi(t,\cdot)+\overline{\phi}\|_{L^\infty},\ 0\le t\le T\} < \infty$ we see that problem (17), (18) is equivalent in $(0,T)$ with the following problem

$$\frac{d}{dt}\binom{y}{\psi} + A\binom{y}{\psi} + F_0\binom{y}{\psi} = \binom{0}{\tau^{-1}\overline{g}}, \tag{23}$$

$$\binom{y}{\psi}(0) = \binom{y_0}{\psi_0}, \tag{24}$$

where

$$F_0\binom{y}{\psi} = \binom{0}{\overline{\beta}_0(\psi+\overline{\phi})},$$

with $\beta_0 : \mathbf{R}\to\mathbf{R}$ a nondecreasing Lipschitz function such that $\beta_0(r) = (1/2a\tau)r^3$ for $|r| \le M$. Therefore F_0 is everywhere defined on $H = L^2(\Omega)^2$, monotone and Lipschitzian. So all the assumptions of Corollary 1 in [1] are satisfied for problem (23), (24) (with $K = H$). The corresponding fractional step scheme associated with (23), (24) is

$$\frac{\partial}{\partial t}\binom{y_\epsilon}{\psi_\epsilon} + A\binom{y_\epsilon}{\psi_\epsilon} = \binom{0}{\tau^{-1}},\ t\in[i\epsilon,(i+1)\epsilon),\ x\in\Omega\,,$$

$$y_\epsilon = 0,\ \psi_\epsilon = 0 \text{ in } [i\epsilon,(i+1)\epsilon)\times\Gamma\,,$$

$$y_\epsilon(i\epsilon+0,x) = y_\epsilon(i\epsilon-0,x),\ \psi_\epsilon(i\epsilon+0,x) = \theta_\epsilon(\epsilon-0,x)\,, \tag{25}$$

$$x\in\Omega,\ i = 1,2,\ldots,N-1,$$

$$y_\epsilon(0,\cdot) = y_0,\ \psi_\epsilon(0,\cdot) = \psi_0\,,$$

$$\frac{\partial\theta_\epsilon}{\partial t} + \beta_0(\theta_\epsilon + \overline{\phi}) = 0 \text{ in } [0,\epsilon]\times\Omega \ ,$$
$$\theta_\epsilon(0,\cdot) = \psi_\epsilon(i\epsilon - 0,\cdot)\ ,\quad i = 1,2,...,N-1\ , \tag{26}$$

where $N = [T/\epsilon]$. According to [1] we have

$$y_\epsilon \to y,\ \psi_\epsilon \to \psi \text{ in } C([0,T];\ L^2(\Omega)) \text{ as } \epsilon \to 0^+\ .$$

Remark 3.1 We see that (25) is a linear partial differential system on each interval $[i\epsilon,(i+1)\epsilon)$ while (26) is an ordinary differential equation. This fact shows the advantage of the fractional step method in the case of phase field model . We recall that to apply the result in [1] we replaced F by F_0. It seems this is not necessary but this is an open problem. Moreover, we believe that the convergence of the fractional step scheme takes place in a stronger topology. Another problem is to establish a rate of convergence for this scheme in the case of phase field model.

Note. The results of this paper were reported in 1992 in Professor Barbu's seminar (University of Iași, Romania).

References

1. V. Barbu and M. Iannelli, Approximating some non-linear equations by a fractional step scheme, *Diff. Int. Equations* 6, 1993, 15-26.

2. P.W. Bates and Songmu Zheng, Inertial manifolds and inertial sets for the phase-field equations, in vol. "*Dynamical Systems*", Plenum, 1992, pp. 375-398.

3. G. Caginalp, An anlysis of a phase field model of a free boundary, *Arch. Rat. Mech. Anal.* 92, 1986, 205-245.

4. G. Cagninalp and Y. Nishiura, The existence of travelling waves for phase field equations and convergence to sharp inteerface models in the singular limit, *Quart. Appl. Math.* 49, 1991, 147-162.

5. C. M. Elliott and Zheng Songmu, Global existence and stability of solutions to the phase field equations, in "*Free boundary problems*", K.H. Hoffmann and J. Sprekels (Eds.), Int. Series of Numerical Math., vol. 95, Birkhäuser, Basel, 1990, pp. 46-58.

6. G. Moroșanu, "*Nonlinear Evolution Equations and Applications*", D. Reidel, Dordrecht-Boston-Lancaster-Tokyo, 1988.