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A new generalization of the integral operators of Kim and Merkes and of Pfaltzgraff

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Abstract

In this note we study an integral operator which preserves the univalence and which generalizes the integral operator of Kim and Merkes and also the integral operator of Pfaltzgraff.

1 Introduction

Let A denote the class of functions f which are analytic in the unit disk $U = \{ z \in C : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$.
Let S denote the class of functions $f \in A$, f univalent in U.

Many authors studyed the problem of integral operators which preserve the class S. In this sense, the results due to Kim and Merkes([2]) and Pfaltzgraff([6]) are well-known.

Theorem 1.1 *([2]). Let $f \in S$, $\beta \in C$. If $|\beta| \le 1/4$, then the function*

(1) $$F(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\beta du$$

is univalent in U.

Theorem 1.2 *([6]). Let $f \in S$, $\delta \in C$. If $|\delta| \le 1/4$, then the function*

(2) $$F(z) = \int_0^z (f'(u))^\delta du$$

is univalent in U.

2 Preliminaries

Lemma 2.1 *([1]). Let $f \in S$. Then*

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+|z|}{1-|z|} \qquad (\forall) z \in U$$

and

$$|-2|z|^2+(1-|z|^2)\frac{zf''(z)}{f'(z)}|\leq 4|z| \qquad (\forall)z\in U$$

Theorem 2.1 *([5]). Let $g\in A$, $\alpha\in C$, $Re\alpha>0$. If*

(3) $$\frac{1-|z|^{2Re\alpha}}{Re\alpha}|\frac{zg''(z)}{g'(z)}|\leq 1, \qquad (\forall)z\in U$$

then the function

$$F_\alpha(z)=\left(\alpha\int_0^z u^{\alpha-1}g'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U.

Theorem 2.2 *([4]). Let $g\in A$. Let α, β and c be complex numbers, $Re\alpha>0$, $Re(\alpha+\beta)>0$, $Re\beta/\alpha>-1/2$, $|c|<1$ and $|c(\alpha+\beta)+\beta|+|\beta|\leq|\alpha+\beta|$. If*

(4) $$|c|z|^{2(\alpha+\beta)}+\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zg''(z)}{g'(z)}-\beta\right)|\leq 1$$

for all $z\in U\setminus\{0\}$, then the function

$$G_\alpha(z)=\left(\alpha\int_0^z u^{\alpha-1}g'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U.

3 Main results

Theorem 3.1 *Let $f\in S$, $n\in N$, α, $\beta\in C$. If*

(5) $$|\beta|\leq\begin{cases} Re\alpha/(2n) & \text{if} \quad Re\alpha\in(0,n/2)\\ 1/4 & \text{if} \quad Re\alpha\in[n/2,\infty)\end{cases}$$

then the function

(6) $$F_{\alpha,\beta,n}(z)=\left(\alpha\int_0^z u^{\alpha-1}\left(\frac{f(u^n)}{u^n}\right)^\beta du\right)^{1/\alpha}$$

is analytic and univalent in U.

Proof. Because the function f is univalent in U, we can choose the analytic branch of $\left(\frac{f(u^n)}{u^n}\right)^\beta$ equal to 1 at the origin and then the function g belongs to A, where

$$g(z)=\int_0^z\left(\frac{f(u^n)}{u^n}\right)^\beta du$$

We have

$$\frac{zg''(z)}{g'(z)} = \beta \cdot n \left(\frac{z^n f'(z^n)}{f(z^n)} - 1 \right)$$

¿From (3) and applying Lemma 2.1, we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha} |\frac{zg''(z)}{g'(z)}| \le \frac{2n|\beta|}{Re\alpha} \cdot \frac{1-|z|^{2Re\alpha}}{1-|z|^n} =$$

$$= \frac{2n|\beta|}{Re\alpha} \frac{1-(|z|^n)^{2Re\alpha/n}}{1-|z|^n}$$

Let us consider the function φ ,$\varphi : [0,1) \longrightarrow R$,

$$\varphi(x) = \frac{1-x^b}{1-x}$$

It is easy to prove that

$$\varphi(x) \le \begin{cases} 1 & \text{if} \quad b \in (0,1) \\ b & \text{if} \quad b \in [1,\infty) \end{cases}$$

Then we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha} \cdot |\frac{zg''(z)}{g'(z)}| \le \begin{cases} 2n|\beta|/Re\alpha & \text{if} \quad Re\alpha \in (0, n/2) \\ 4|\beta| & \text{if} \quad Re\alpha \ge n/2 \end{cases}$$

In view of (5), the inequality (3) is true and from Theorem 2.1 we conclud that the function

$$F_{\alpha,\beta,n}(z) = \left(\alpha \int_0^z u^{\alpha-1} g'(u) du \right)^{1/\alpha} =$$

$$= \left(\alpha \int_0^z u^{\alpha-1} \left(\frac{f(u^n)}{u^n} \right)^\beta du \right)^{1/\alpha}$$

is analytic and univalent in U.

The following results are proved by using Theorem 2.2 in the particular case $\beta = n - \alpha$, where $n \in N$. For this choice, from Theorem 2.2 we get the following

Corollary 3.1 *Let $g \in A$. Let α, c be complex numbers and let n be a positive integer number. If $|\alpha - n| < n$, $|c| < 1$, $|cn + n - \alpha| + |n - \alpha| \le n$ and*

$$(7) \quad |c|z|^{2n} + \frac{1-|z|^{2n}}{n} \left(\frac{zg''(z)}{g'(z)} + \alpha - n \right)| \le 1$$

for all $z \in U$, then the function

$$G_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} g'(u) du \right)^{1/\alpha}$$

is analytic and univalent in U.

Theorem 3.2 *Let $f \in S$, $n \in N$, α, β, $\delta \in C$. If $|\alpha - n| < n$, $|\beta| + |\delta| \leq (n - |\alpha - n|)/(4n)$,then the function*

$$(8) \quad F_{\alpha,\beta,\delta,n}(z) = \left(\alpha \int_0^z u^{\alpha-1} \left(\frac{f(u^n)}{u^n}\right)^{\beta} (f'(u^n))^{\delta}\, du\right)^{1/\alpha}$$

is analytic and univalent in U.

Proof. Because the function f is univalent in U, we can choose the analytic branch of $\left(\frac{f(u^n)}{u^n}\right)^{\beta}$ equal to 1 at the origin and also the analytic branch of $(f'(u^n))^{\delta}$ equal to 1 at the origin and then the function g belongs to A, where

$$g(z) = \int_0^z \left(\frac{f(u^n)}{u^n}\right)^{\beta} (f'(u^n))^{\delta}\, du$$

We have

$$(9) \quad \frac{zg''(z)}{g'(z)} = \beta n \left(\frac{z^n f'(z^n)}{f(z^n)} - 1\right) + \delta n \frac{z^n f''(z^n)}{f'(z^n)}$$

In view of (9), from (7) we get

$$| c|z|^{2n} + \frac{1-|z|^{2n}}{n}(\frac{zg''(z)}{g'(z)} + \alpha - n)| =$$

$$= | \delta \left(-2|z|^{2n} + (1-|z|^{2n}) \frac{z^n f''(z^n)}{f'(z^n)}\right) + \beta (1-|z|^{2n}) \left(\frac{z^n f'(z^n)}{f(z^n)} - 1\right) +$$

$$+ (c + 2\delta + 1 - \alpha/n)|z|^{2n} + (\alpha - n)/n|$$

If $c = -2\delta - 1 + \alpha/n$, from $|\alpha - n| < n$ and $|\delta| \leq (n - |\alpha - n|)/(4n)$ it results that $|c| < 1$ and also

$$|cn + n - \alpha| + |n - \alpha| = |2n\delta| + |n - \alpha| < n$$

Using Lemma 2.1 and in view of assertion $|\beta| + |\delta| \leq (n - |\alpha - n|)/(4n)$ it follows that

$$| c|z|^{2n} + \frac{1-|z|^{2n}}{n}(\frac{zg''(z)}{g'(z)} + \alpha - n)| \leq 4|\delta| + 4|\beta| + \frac{|\alpha - n|}{n} \leq 1$$

¿From Corollary 3.1 we conclud that the function $F_{\alpha,\beta,\delta,n}$ defined by (8) is analytic and univalent in U.

Remarks

1. For $\alpha = 1, n = 1$ and $\delta = 0$ from Theorem 3.2 we get Theorem 1.1.
2. For $\alpha = 1, n = 1$ and $\beta = 0$ from Theorem 3.2 we refind Theorem 1.1.
3. If in Theorem 3.2 we take $n = 1$ we deduce the following

Corollary 3.2 *Let $f \in S$, α, β, $\delta \in C$. If $|\alpha-1| < 1$, $|\beta|+|\delta| \leq (1-|\alpha-1|)/4$, then the function*

$$F(z) = \left(\alpha \int_0^z u^{\alpha-1} \left(\frac{f(u)}{u} \right)^{\beta} (f'(u))^{\delta} \, du \right)^{1/\alpha}$$

is analytic and univalent in U.

We note that this result was obtained in [3] using another condition for univalence.

4. In the particular case $\delta = 0$, as a direct consequence of Theorem 3.2 ,we obtain the following

Theorem 3.3 *Let $f \in S$, $n \in N$, α, $\beta \in C$. If $|\alpha - n| < n$, $|\beta| \leq (n - |\alpha - n|)/(4n)$, then the function*

$$F_{\alpha;\beta;n}(z) = \left(\alpha \int_0^z u^{\alpha-1} \left(\frac{f(u^n)}{u^n} \right)^{\beta} du \right)^{1/\alpha}$$

is analytic and univalent in U.

Remark. In this case we reach the same conclusion as the one from Theorem 3.1, but this result is not so good as the one from Theorem 3.1 .

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