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A Sharp Sufficient Condition for a Subclass of Starlike Functions

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Abstract. If f is an analytic function in the unit disc U, with $f(0)=0$, $f'(0)\neq 0$ and

$$\left|1+\frac{zf''(z)}{f'(z)}\right| < M_0 = 2.23734..., \text{ for } z\in U,$$

where M_0 is given by the system (11), then

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1, \text{ for } z\in U,$$

and this result is sharp.

1. INTRODUCTION.

In [2] the following sufficient condition for starlikeness was obtained. If f is an analytic function in the unit disc U, with $f(0)=0$, $f'(0)\neq 0$ and

$$\left|1+\frac{zf''(z)}{f'(z)}\right| < 2, \text{ for } z\in U,$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1, \text{ for } z\in U.$$

A natural question is to find the maximum value of M, for which the inequality

$$\left|1+\frac{zf''(z)}{f'(z)}\right| < M, \text{ for } z\in U, \tag{1}$$

implies

$$\left|\frac{z\,f'(z)}{f(z)} - 1\right| < 1, \quad z \in U. \tag{2}$$

i.e. f belongs to a subclass of starlike functions.

In this paper, using a special result on differential subordinations, we give the definitive answer to the above problem, by finding the biggest $M = M_0 = 2.23734...$ such that (1) implies (2). In particular in (1) we can take $M = \sqrt{5} = 2.23606...$ which is a good lower approximation of M_0.

Note that in [1] we found the biggest $M = M_1 = 2.84116...$ such that (1) implies the starlikeness of f.

2 PRELIMINARIES.

An analytic function f in U, with $f(0) = 0$, is said to be *starlike* if it is univalent and $f(U)$ is a starlike domain with respect to the origin. It is well known that f is starlike if and only if $f(0) = 0$, $f'(0) \neq 0$ and

$$\mathrm{Re}\frac{z\,f'(z)}{f(z)} > 0, \text{ for } z \in U.$$

A special subclass of starlike functions, intensively studied by many authors, consists of functions f, $f(0) = 0$, $f'(0) \neq 0$, which satisfies the condition (2), which is equivalent to

$$\mathrm{Re}\frac{f(z)}{z\,f'(z)} > \frac{1}{2}, \text{ for } z \in U,$$

If f and g are analytic functions in U and g is univalent, then f is *subordinate* to g, written $f \prec g$, or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We will need the following subordination result, which is a particular case of a more general result concerning the Briot-Bouquet differential subordinations and can be easily obtained by combining Corollary 1.1 and Theorem 5 in [3].

LEMMA. *Let h be analytic in U, with $h(0) = 1$ and let q, with $q(0) = 1$, satisfy the differential equation*

$$q(z) + \frac{z\,q'(z)}{q(z)} = h(z). \tag{3}$$

If we suppose that

(a) h *is convex in* U,

(b) $\operatorname{Re} q(z) > 0$, *for* $z \in U$,

then q *is univalent and is given by*

$$q(z) = \frac{z\,k'(z)}{k(z)}, \tag{4}$$

where

$$k(z) = \int_0^z \frac{g(t)}{t}\,\mathrm{dt} \tag{5}$$

and

$$g(z) = z \exp \int_0^z \frac{h(t)-1}{t}\,\mathrm{dt}. \tag{6}$$

Moreover if f *is analytic in* U, *with* $f(0) = 0$, $f'(0) \neq 0$ *and*

$$1 + \frac{z\,f''(z)}{f'(z)} \prec h(z)$$

then

$$\frac{z\,f'(z)}{f(z)} \prec q(z). \tag{7}$$

This result is sharp and the extremal function is $f = k$.

3. MAIN RESULT.

In order to state our main result we shall use the following notations:

$$\varphi(M,t) = 1 + \frac{2}{M}\cdot\cos t + \frac{1}{M^2}, \tag{8}$$

$$\psi(M,t) = (M^2-1)\cdot\arctan\frac{\sin t}{M+\cos t} + t. \tag{9}$$

and

$$\chi(M,t) = (M + \frac{1}{M})\cos t + 2. \tag{10}$$

Theorem. *If f is an analytic function in U, with $f(0)=0$, $f'(0)\neq 0$ and*

$$\left|1+\frac{z\,f''(z)}{f'(z)}\right| < M_0 = 2.23734\ldots, \text{ for } z\in U,$$

where M_0 is the smallest value of $M>\sqrt{5}$, where (M,t) satisfies the system:

$$\begin{cases} \frac{1}{M}\cos t + \frac{1}{M^2} - \frac{1}{M}[\varphi(M,t)]^{(1-M^2)/2}\cdot\cos\psi(M,t) - \frac{1}{2} = 0 \\ -\sin t - \{\frac{M^2-1}{M}\sin t\cos\psi(M,t) - \chi(M,t)\cdot\sin\psi(M,t)\}\cdot[\varphi(M,t)]^{-(1+M^2)/2} = 0, \end{cases} \tag{11}$$

then

$$\left|\frac{z\,f'(z)}{f(z)}-1\right| < 1, \text{ for } z\in U$$

and this result is sharp.

Proof. The inequality (1) can be rewritten as

$$1+\frac{z\,f''(z)}{f'(z)} \prec M\,\frac{Mz+1}{z+M},\ z\in U.$$

If we let $h(z)=M(Mz+1)/(z+M)$, then from (6) and (5) we obtain

$$g(z) = z\,(1+\frac{z}{M})^{M^2-1}$$

and

$$k(z) = \frac{1}{M}\,[(1+\frac{z}{M})^{M^2}-1]$$

respectively. Hence, according to (4), the solution q of the differential equation (3) is given by

$$q(z) = q_M(z) = \frac{Mz(1+z/M)^{M^2-1}}{(1+z/M)^{M^2}-1},\ z\in U. \tag{12}$$

Since h is convex, in order to apply the above lemma we only need to put the condition $\operatorname{Re} 1/q(z) > 1/2$ in U. In this case from (7) we deduce that f satisfies $|z\,f'/f-1|<1$. Therefore our problem is to find the maximum value of M such that

$$\operatorname{Re}\frac{1}{q_M(z)} > 1/2, \quad \text{for } |z|\le 1, \tag{13}$$

where $q_M(z)$ is given by (12).

We first prove that (13) holds for $M = \sqrt{5} = 2.23606\ldots$, which will be a good lower approximation of M_0.

For $z = x + i\sqrt{1-x^2}$, $x \in [-1,1]$ and $M = \sqrt{5}$, (13) becomes

$$\mathrm{Re}\frac{1}{q_M(z)} - 1/2 = \frac{50x^4 + 95\sqrt{5}x^3 + 290x^2 + 51\sqrt{5}x + 14}{20(\sqrt{5}x+3)^4},\ x \in [-1,1].$$

We shall prove that

$$P(x) = 50x^4 + 95\sqrt{5}x^3 + 290x^2 + 51\sqrt{5}x + 14 > 0, \tag{14}$$

for $x \in [-1,1]$. If we let $x = y/\sqrt{5}$ then (14) becomes

$$Q(y) = 2y^4 + 19y^3 + 58y^2 + 51y + 14 > 0,\ \text{ for } y \in [-\sqrt{5}, \sqrt{5}].$$

The roots of the equation $Q'(y) = 0$ are $y_1 = -3$, $y_{2,3} = -33/16 \pm \sqrt{545}/16$. The only root in the interval $[-\sqrt{5}, \sqrt{5}]$ is $y_3 = -33/16 + \sqrt{545}/16$. Since $Q(-\sqrt{5}) = 354 - 146\sqrt{5} \approx 27.534 > 0$, $Q(\sqrt{5}) = 354 + 146\sqrt{5} > 0$, and $Q(y_3) \approx 0.43487\ldots > 0$ we easyly deduce that $Q(y) > 0$ for $y \in [-\sqrt{5}, \sqrt{5}]$.

We now deduce the system (11) which will give the exact bound $M_0 = 2.237349\ldots$. If we let $z = e^{it}$, $t \in [-\pi, \pi]$ then the inequality (13) is equivalent to $F(M,t) \geq 0$ where

$$F(M,t) = \frac{1}{M}\cos t + \frac{1}{M^2} - \frac{1}{M}\varphi(M,t)^{(1-M^2)/2}\cos\psi(M,t) - \frac{1}{2} \tag{15}$$

$t \in [-\pi, \pi]$, $M > \sqrt{5}$. Hence M_0 will be obtained as the smallest $M > \sqrt{5}$ such that (M,t) satisfies the system

$$\begin{cases} F(M,t) = 0 \\ \partial/\partial t\ F(M,t) = 0. \end{cases}$$

According to (15) this system becomes (11).

By Newton's method we obtain the following approximate solution of the system: $(M_0, t_0) = (2.2373496, 1.8397920)$. Using the computer (with 30 digits precision) we find that

$$\min_{t\in[0,\pi]} F(2.2373496, t) \approx 1.018949\ldots \cdot 10^{-9} > 0.$$

On the other hand

$$F(2.23734961, 1.8397920) \approx -4.074325\ldots \cdot 10^{-9} < 0.$$

This shows that the first 7 digits of $M_0 = 2.23734960$ are exact.

The images of the unit disc U by the extremal functions q and k for $M = M_0$ are given in the following figure.

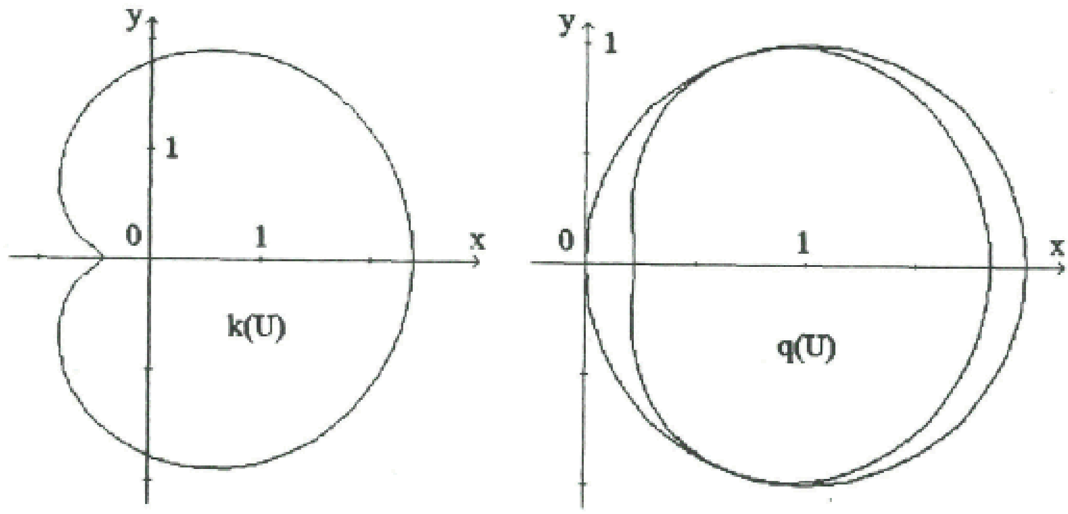

FIGURE 1

REFERENCES

[1] V. Anisiu, P.T.Mocanu and I. Şerb, A sharp criterion for starlikeness (preprint).

[2] S.S. Miller and P.T. Mocanu, On some classes of first-order differential subordinations, *Michigan Math. J.* **32** (1985), 185-195.

[3] S.S. Miller and P.T. Mocanu, A special differential subordination and its application to univalency conditions, *Current Topics in Analytic Function Theory*, World Scientific, Singapore, London (1992), 171-185.