LIBERTAS MATHEMATICA, VOL. XVI (1996)

# An Extension of Jack's-Miller's-Mocanu's Lemma for Holomorphic Mappings Defined on Some Domains in $C^n$

**GABRIELA KOHR CORNEL PINTEA**

In this paper the authors obtain an extension of Jack's-Miller's-Mocanu's Lemma and some applications are considered.

*AMS No. 32H02, 30C45*

## 1. INTRODUCTION

In several papers S.S Miller and P.T. Mocanu used the analytic functions defined on the unit disc, which satisfy some differential inequalities and obtained several inclusion relations, inequalities and some sufficient conditions for univalence, see [6], [7].

P.Liczberski [5] and also, G. Kohr and M. Kohr-Ile [3] obtained some results concerning partial differential subordinations for holomorphic mappings defined on the unit Euclidian ball and the unit polydisc, respectively.

Very recently G. Kohr and P. Liczberski [4] considered the holomorphic mappings defined on the unit ball in $\mathbf{C}^n$ with an arbitrarily fixed norm, and gave some properties for these mappings, concerning inclusions relations or subodinations.

These results are based on several versions of Jack-Miller-Mocanu Lemma [6] and in this paper we find a new generalization of this result for holomorphic mappings defined on some domains in $\mathbf{C}^n$.

## 2. PRELIMINARIES

We let $\mathbf{C}^n$ denote the space of n complex variables

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

with the Euclidian inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j$ and the norm $||z|| = \sqrt{\langle z, z\rangle}$. The open Euclidian ball $\{z \in \mathbf{C}^n \ : \ ||z|| < r\}$ is denoted by $B_r$, the open unit Euclidian ball is abbreviated by $B_1 = B$. The origin

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

is always denoted by 0. As usual by $L(\mathbf{C}^n, \mathbf{C}^m)$ we denote the space of all continuous linear operators from $\mathbf{C}^n$ into $\mathbf{C}^m$ with the standard operator norm. The letter I will always represent the identity operator in $L(\mathbf{C}^n, \mathbf{C}^n)$. The class of holomorphic mappings from a domain $G \subseteq \mathbf{C}^n$ into $\mathbf{C}^n$ is denoted by $H(G)$. A mapping $f \in H(G)$ is said to be locally biholomorphic in $G$ if its Fréchét derivative $Df(z) = (\frac{\partial f_j}{\partial z_k}(z))_{1\leq j,k\leq n}$ as an element of $L(\mathbf{C}^n, \mathbf{C}^n)$ is nonsingular at each point $z \in G$ (or, equivalently, if it has a locally holomorphic inverse at any point $z \in G$). A mapping $f \in H(G)$ is called biholomorphic if the inverse mapping $f^{-1}$ does exist, is holomorphic on a domain $\Omega$ and $f^{-1}(\Omega) = G$. If $D^2f(z)$ means the Fréchét derivative of the second order of $f \in H(G)$ at the point $z$, then of course $D^2f(z)$ is a continuous biliniar operator from $\mathbf{C}^n \times \mathbf{C}^n$ into $\mathbf{C}^n$ and its restriction $D^2f(z)(u,\cdot)$ to $u \times \mathbf{C}^n$ belongs to $L(\mathbf{C}^n, \mathbf{C}^n)$.

For a set $G \subseteq \mathbf{C}^n$ the closure and the topological bord will be denoted by $\overline{G}$ and $\partial G$.

If $g$ is a $C^2$ real valued function defined on a domain $\Omega \subseteq \mathbf{C}^n$, we denote by

$$\frac{\partial g}{\partial z}(z) = (\frac{\partial g}{\partial z_1}(z), \cdots, \frac{\partial g}{\partial z_n}(z))'$$

$$\frac{\partial^2 g}{\partial z^2}(z) = (\frac{\partial^2 g}{\partial z_i \partial z_j}(z))_{1 \leq i,j \leq n} \text{ and } \frac{\partial^2 g}{\partial \bar{z} \partial z}(z) = (\frac{\partial^2 g}{\partial \bar{z}_i \partial z_j}(z))_{1 \leq i,j \leq n}$$

for all $z \in \Omega$, where the prime "$'$" means the transpose of elements and matrix defined on $\mathbf{C}^n$.

We say that $z_0$ is a critical point for $g$ if $\frac{\partial g}{\partial z}(z_0) = 0$. We denote by $\mathcal{C}(g)$ the set of critical points of the function $g$. We also call a number $t \in \mathbf{R}$ regular value of $g$, if either $g^{-1}(t) = \emptyset$ or $g^{-1}(t) \cap \mathcal{C}(g) = \emptyset$.

LEMMA 2.1.([6],[7]) *Let $f : U = \{z \in \mathbf{C} : |z| < 1\} \to \mathbf{C}$ be a holomorphic function such that $f(0) = 0$ and $f(z) \not\equiv 0$, $z \in U$. Suppose that at $z_0 \in U$ the following condition is satisfied*

$$|f(z_0)| = \max\{|f(z)| \; : \; |z| \leq |z_0|\},$$

*Then there exists a real number $m \geq 1$ such that*

$$z_0 f'(z_0) = m f(z_0)$$

$$Re[1 + \frac{z_0 f''(z_0)}{f'(z_0)}] \geq m.$$

LEMMA 2.2. ([5]) *Let $f : B \to \mathbf{C}^n$ be a holomorphic mapping with $f(0) = 0$. Suppose that at $z_0 \in B$*

$$\| f(z_0 \| = \max\{\| f(z) \| : \| z \| \leq \| z_0 \|\}$$

*then there exists a real number $m \geq 1$ such that*

$$Re\langle Df(z_0)(z_0), f(z_0)\rangle = m \| f(z_0 \|^2$$

$$Re\langle D^2 f(z_0)(z_0, z_0), f(z_0)\rangle \geq m(m-1) \parallel f(z_0 \parallel^2 .$$

## 3. MAIN RESULTS

Let $D$ be a domain in $\mathbf{C}^n$ whose border $\partial D$ is a $C^2$ real hypersurface which can be represented under the form

$$\partial D = \varphi^{-1}(t),$$

where $t$ is a regular value of a $C^2$ differentiable function $\varphi$ defined on a neighborhood of $\partial D$.

THEOREM 3.1. *Let $D$ be a domain in $\mathbf{C}^n$ satisfying the above conditions with $0 \in D$, $z_0 \in \partial D$ and let $f \in H(D \cup \{z_0\}) \cap C(\overline{D})$ such that $f(0) = 0$, $f \not\equiv 0$ and $f$ is locally biholomorphic at $z_0$. If $\phi \in C^2(V, \mathbf{R})$, where $V$ is a neighborhood of $f(\partial D)$, such that $z_0 \in \mathcal{C}(\phi \circ f|_{\partial D})$ and $\frac{\partial \phi}{\partial w}(w_0) \neq 0$, where $w_0 = f(z_0)$, then there exists a real nonzero number $s$ such that the following condition is satisfied*

$$([Df(z_0)]^{-1})' \frac{\partial \varphi}{\partial z}(z_0) = s \frac{\partial \phi}{\partial w}(w_0) \tag{3.1}$$

*Proof.* Since $f$ is locally biholomorphic at $z_0$, there exists $W$ an open neighborhood of $z_0$ such that $f : W \to f(W)$ is a diffeomorphism. If $T_{z_0}(\partial D \cap W)$ means the real tangent space at $\partial D \cap W$ in the point $z_0$ and if $v \in T_{z_0}(\partial D \cap W)$ is an arbitrary tangent vector to $\partial D \cap W$ at $z_0$, then it is well known that there exist $\varepsilon > 0$ and $\gamma : (-\varepsilon, \varepsilon) \to \partial D \cap W$ a twice differentiable curve such that $\gamma(0) = z_0$ and $\frac{d\gamma}{dt}(0) = v$. Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbf{R}$ be the function defined by $\alpha(t) = (\phi \circ f \circ \gamma)(t)$, $t \in (-\varepsilon, \varepsilon)$. Because $z_0 \in \mathcal{C}(\phi \circ f|_{\partial D})$ it follows that $d(\phi \circ f|_{\partial D})_{z_0} = 0$ which means that $d(\phi \circ f|_{\partial D})_{z_0}(v) = 0$, or else $d(\phi \circ f|_{\partial D})_{\gamma(0)}(\frac{d\gamma}{dt}(0)) = 0$, namely $\alpha'(0) = 0$.

On the other hand,

$$\alpha'(0) = \frac{d}{dt}(\phi \circ f \circ \gamma)(t)|_{t=0} = \sum_{j=1}^{n} \frac{\partial \phi}{\partial w_j}(w_0) \sum_{i=1}^{n} \frac{\partial f_j}{\partial z_i}(z_0) \frac{d\gamma_i}{dt}(0) +$$

$$+ \sum_{j=1}^{n} \frac{\partial \phi}{\partial \overline{w}_j}(w_0) \sum_{i=1}^{n} \frac{\partial \overline{f}_j}{\partial \overline{z}_i}(z_0) \frac{d\overline{\gamma}_i}{dt}(0) = 2Re \sum_{j=1}^{n} \frac{\partial \phi}{\partial w_j}(w_0) \sum_{i=1}^{n} \frac{\partial f_j}{\partial z_i}(z_0) \frac{d\gamma_i}{dt}(0),$$

hence we get that

$$\alpha'(0) = 2Re\langle Df(z_0)(v), \overline{\tfrac{\partial \phi}{\partial w}}(w_0)\rangle, \tag{3.2}$$

$$\text{so , } Re\langle Df(z_0)(v), \overline{\frac{\partial \phi}{\partial w}}(w_0)\rangle = 0$$

If we denote by $\langle \frac{\partial \phi}{\partial \overline{w}}(w_0)\rangle^{\perp}$ the real orthogonal complement of $\frac{\partial \phi}{\partial \overline{w}}(w_0)$, we obtain that $Df(z_0)v \in \langle \frac{\partial \phi}{\partial \overline{w}}(w_0)\rangle^{\perp}$, so $Df(z_0)(T_{z_0}(\partial D \cap W)) \subseteq \langle \frac{\partial \phi}{\partial \overline{w}}(w_0)\rangle^{\perp}$.
Since $dim_{\mathbf{R}} Df(z_0)(T_{z_0}(\partial D \cap W)) = dim_{\mathbf{R}} \langle \frac{\partial \phi}{\partial \overline{w}}(w_0)\rangle^{\perp}$, we conclude that

$$\langle \tfrac{\partial \phi}{\partial \overline{w}}(w_0)\rangle^{\perp} = Df(z_0)(T_{z_0}(\partial D \cap W)) \tag{3.3}$$

On the other hand $f$ is locally biholomorphic at $z_0$ then easily $f(\partial D \cap W)$ is a real hypersurface and

$$Df(z_0)(T_{z_0}(\partial D \cap W)) = T_{f(z_0)} f(\partial D \cap W) \tag{3.4}$$

where $T_{f(z_0)} f(\partial D \cap W)$ is the real tangent space of $f(\partial D \cap W)$ at $f(z_0)$.
Now, using the conditions (3.3) and (3.4), we get the following relation

$$\langle \tfrac{\partial \phi}{\partial \overline{w}}(w_0)\rangle^{\perp} = T_{f(z_0)} f(\partial D \cap W) \tag{3.5}$$

If $g : f(\partial D \cap W) \to \mathbf{R}$ is given by

$$g(w) = \varphi(f^{-1}(w)),$$

for all $w \in f(\partial D \cap W)$, then $\frac{\partial g}{\partial \overline{w}}(w_0)$ is a normal vector to $f(\partial D \cap W)$ at $w_0$.

Using similar kind of arguments as in the proof of the relation (3.3), and also the condition (3.5), we conclude that

$$\langle \frac{\partial g}{\partial \overline{w}}(w_0)\rangle^{\perp} = \langle \frac{\partial \phi}{\partial \overline{w}}(w_0)\rangle^{\perp},$$

so, we can find a real number $s$ with

$$\frac{\partial g}{\partial \overline{w}}(w_0) = s\frac{\partial \phi}{\partial \overline{w}}(w_0).$$

On the other hand, since

$$\frac{\partial g}{\partial w_j}(w_0) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial z_i}(z_0)\frac{\partial z_i}{\partial w_j}(w_0),$$

this implies that

$$\frac{\partial g}{\partial w}(w_0) = [Df^{-1}(w_0)]'\frac{\partial \varphi}{\partial z}(z_0),$$

which shows the condition (3.1). It remains only to prove that $s \neq 0$. Indeed, because $f$ is locally biholomorphic at $z_0$, $\frac{\partial \phi}{\partial w}(w_0) \neq 0$ and $\frac{\partial \varphi}{\partial z}(z_0)$, then it is clear that $s \neq 0$.

*Remark 3.1.* Using the proof of Theorem (3.1) we conclude that $\frac{\partial \phi}{\partial \overline{w}}(w_0)$ is coliniar with the normal vector to $f(\partial D \cap W)$ at $f(z_0)$ i.e. the level hypersurfaces of $\phi$ which pass through $f(z_0)$ and $f(\partial D \cap W)$ are tangent in $f(z_0)$.

THEOREM 3.2. *Under the conditions of Theorem 3.1, if $z_0$ is particularly a point of local maximum of the function $\phi \circ f|_{\partial D}$, then the condition (3.1) is obviously satisfied, and furthermore the following condition holds*

$$Re\{(Df(z_0)v)'\frac{\partial^2 \phi}{\partial w^2}(w_0)Df(z_0)v\} + (\overline{Df(z_0)v})'\frac{\partial^2 \phi}{\partial \overline{w}\partial w}(w_0)Df(z_0)v+$$
$$+Re\{(\tfrac{\partial \phi}{\partial w}(w_0))'D^2 f(z_0)(v,v)\} \leq \tfrac{1}{s}Re\{v'\tfrac{\partial^2 \varphi}{\partial z^2}(z_0)v\} + \tfrac{1}{s}v'\tfrac{\partial^2 \varphi}{\partial \overline{z}\partial z}(z_0)v, \qquad (3.6)$$

*for all* $v \in \mathbf{C}^n \setminus \{0\}$ with $Re\langle\frac{\partial \varphi}{\partial z}(z_0), \overline{v}\rangle = 0$.

*Proof.* If $\alpha$, $\gamma$ are the mappings defined as in the proof of Theorem 3.1, which correspond to the vector $v \in T_{z_0}(\partial D)$, then it is clear that $\alpha'(0) = 0$ and $\alpha''(0) \leq 0$.

Since

$$\alpha''(0) = 2Re \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial w_i \partial w_j}(w_0)\frac{dw_i}{dt}(0)\frac{dw_j}{dt}(0) + 2\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial w_i \partial \overline{w}_j}(w_0)\frac{dw_i}{dt}(0)\frac{d\overline{w}_j}{dt}(0) +$$

$$+2Re \sum_{i,j,k=1}^{n} \frac{\partial \phi}{\partial w_j}(w_0)\frac{\partial^2 f_j}{\partial z_i \partial z_k}(z_0)\frac{d\gamma_i}{dt}(0)\frac{d\gamma_k}{dt}(0) + 2Re \sum_{i,j=1}^{n} \frac{\partial \phi}{\partial w_j}(w_0)\frac{\partial f_j}{\partial z_i}(z_0)\frac{d^2\gamma_i}{dt^2}(0),$$

where $w(t) = f(\gamma(t))$, $t \in (-\varepsilon, \varepsilon)$, then we deduce that

$$\alpha''(0) = 2[Re\{(Df(z_0)v)'\frac{\partial^2 \phi}{\partial w^2}(w_0)Df(z_0)v\} + (\overline{Df(z_0)v})'\frac{\partial^2 \phi}{\partial \overline{w}\partial w}(w_0)Df(z_0)v +$$

$$+Re\{(\tfrac{\partial \phi}{\partial w}(w_0))'D^2 f(z_0)(v,v)\} + Re\{(\tfrac{\partial \phi}{\partial w}(w_0))'Df(z_0)(\tfrac{d^2\gamma}{dt^2}(0))\}] \qquad (3.7)$$

On the other hand, the function $\psi : (-\varepsilon, \varepsilon) \to \mathbf{R}$, $\psi = \varphi \circ \gamma$ , is constant on $(-\varepsilon, \varepsilon)$, so $\frac{d\psi}{dt} \equiv 0$ and $\frac{d^2\psi}{dt^2} \equiv 0$ on $(-\varepsilon, \varepsilon)$.

Straightforwaard calculations yields the following relation

$$Re(\tfrac{\partial \varphi}{\partial z}(z_0))'\tfrac{d^2\gamma}{dt^2}(0) = -Re\ v'\tfrac{\partial^2 \varphi}{\partial z^2}(z_0)v - \overline{v}'\tfrac{\partial^2 \varphi}{\partial \overline{z}\partial z}(z_0)v. \qquad (3.8)$$

Now, using the above equality and from (3.7) we obtain

$$\alpha''(0) = 2[Re\{(Df(z_0)v)'\frac{\partial^2 \phi}{\partial w^2}(w_0)Df(z_0)v\} + (\overline{Df(z_0)v})'\frac{\partial^2 \phi}{\partial \bar{w}\partial w}(w_0)Df(z_0)v +$$

$$+Re\{(\frac{\partial \phi}{\partial w}(w_0))'D^2 f(z_0)(v,v)\} - \frac{1}{s}Re\{v'\frac{\partial \varphi}{\partial z^2}(z_0)v\} - \frac{1}{s}\bar{v}'\frac{\partial^2 \varphi}{\partial \bar{z}\partial z}(z_0)v],$$

which yields the relation (3.6). It remains only to prove that $Re\langle \frac{\partial \varphi}{\partial z}(z_0), \overline{v}\rangle = 0$. Indeed, since $v \in T_{z_0}(\partial D)$, the total differential of $\varphi$ at $z_0$ is zero, hence

$$d\varphi_{z_0}(v) = (\frac{\partial \varphi}{\partial z}(z_0))'v + (\frac{\partial \varphi}{\partial \overline{z}}(z_0))'\overline{v} = 2Re\{(\frac{\partial \varphi}{\partial z}(z_0))'v\} = 0$$

If $D$ is the open Euclidian ball $B_r$, for some $r \in (0,1)$ and if we chose $\phi(w) \equiv \| w \|^2$ then, using the result of THEOREM 3.2., we obtain

COROLLARY 3.1. *Let $f \in H(B)$ with $f(0) = 0$ and $f \not\equiv 0$. If there exists $z_0 \in B$ such that is locally biholomorphic at $z_0$ and*

$$\| f(z_0) \| = \max\{\| f(z) \| : \| z \| \leq \| z_0 \|\},$$

*then there exists a real nonzero number $s$, with*

$$0 < s \leq \frac{||z_0||^2}{||f(z_0)||^2} \tag{3.9}$$

*and the following conditions hold*

$$([Df(z_0)]^{-1})'\bar{z}_0 = s\overline{f(z_0)} \tag{3.10}$$

$||v||^2 - Re\{\bar{z}_0'[Df(z_0)]^{-1}D^2f(z_0)(v,v)\} \geq s||Df(z_0)v||^2$, for all $v \in \mathbf{C}^n \setminus \{0\}$ with $Re\langle z_0, v\rangle = 0$. (3.11)

*Proof.* We shall apply the THEOREM 3.2 for the function $\varphi(z) \equiv ||z||^2 - ||z_0||^2$. In this particular case we have $\frac{\partial^2\varphi}{\partial z^2}(z) = O$ and $\frac{\partial^2\varphi}{\partial\bar{z}\partial z}(z) = I$, for all $z \in \mathbf{C}^n$. Further on, using the THEOREM 3.2, the relations (3.10) and (3.11) follow immediately. It remains only to show the relation (3.9).

Since $f$ is locally biholomorphic at $z_0$, the relation (3.10) can be written as follows

$$\bar{z}_0 = s[Df(z_0)]'\overline{f(z_0)} \tag{3.12}$$

On the other hand, using the LEMMA 2.2, there exists a real number $m$, with $m \geq 1$, such that

$$Re\{\overline{f(z_0)}'Df(z_0)z_0\} = m\,||f(z_0)||^2 \tag{3.13}$$

Replacing $f(z_0)$ from (3.12) into (3.13), on can easily get that

$$\frac{1}{s}||z_0||^2 = m\,||f(z_0)||^2,$$

that is $s = \frac{||z_0||^2}{m\,||f(z_0)||^2}$, which yields the inequality (3.9).

## 4. APPLICATIONS

Let $D$ be the domain in $\mathbf{C}^n$ defined by $D = \{z \in \mathbf{C}^n \ : \ \varphi(z) < 1\}$, where $\varphi(z) = \sum_{k=1}^{n} |z_k|^{2p}$ and $p$ is a real number, with $p \geq 1$. Denote $\varphi^{1/2p}(z)$ by $\| z \|_{2p}$ and the domain $D$ by $B_{2p}$. Let $M > 0$ and $\Omega \subseteq \mathbf{C}^n \times \mathbf{C}^n$ be a domain such that $(0,0) \in \Omega$ and $\bigcup_{s \in \mathbf{R}\setminus\{0\}} X_s(M) \subseteq \Omega$, where

$$X_s(M) = \{(u,v) \in \mathbf{C}^n \times \mathbf{C}^n \ : \| u \|_{2p} = M, \ v = ps(|u_1|^{2(p-1)}\bar{u}_1, \ldots, |u_n|^{2(p-1)}\bar{u}_n)'\}$$

Let $Y(\Omega, M)$ be the class of those continuous mappings $h : \Omega \to \mathbf{C}^n$ which satisfy

$$\| h(0,0) \|_{2p} < M \text{ and } \| h(u,v) \|_{2p} \geq M, \text{ for all } (u,v) \in \bigcup_{s \in \mathbf{R}\setminus\{0\}} X_s(M).$$

Using the result of THEOREM 3.2, we obtain:

THEOREM 4.1 *Let $f$ be locally biholomorphic in $B_{2p}$ with $f(0) = 0$ and*

$$(f(z), ([Df(z)]^{-1})'(p|z_1|^{2(p-1)}\bar{z}_1, \ldots, p|z_n|^{2(p-1)}\bar{z}_n)') \in \Omega.$$

*If $h \in Y(\Omega, M)$ is such that*

$$\| h(f(z), ([Df(z)]^{-1})'(p|z_1|^{2(p-1)}\bar{z}_1, \ldots, p|z_n|^{2(p-1)}\bar{z}_n)') \|_{2p} < M$$

*for all $z \in B_{2p}$, then $\| f(z) \|_{2p} < M$ in $B_{2p}$.*

*Proof.* Supposing that the relation $\| f(z) \|_{2p} < M$ is not satisfied over the whole ball $B_{2p}$, then, using the hypothesis $f(0) = 0$ and the continuity of $\varphi$ above defined, we conclude that there exist $0 < r_0 < 1$ and $z_0 \in B_{2p}$, with $\| z_0 \|_{2p} \leq r_0$, such that

$$M = \| f(z_0) \|_{2p} = \max\{\| f(z) \|_{2p} : \| z \|_{2p} \leq r_0\},$$

and hence applying result of the THEOREM 3.2, with $\phi(z) \equiv \varphi(z)$ one can find a real number $s \neq 0$ such that

$$([Df(z)]^{-1})'(p|z_1^0|^{2(p-1)}\bar{z}_1^0, \ldots, p|z_n^0|^{2(p-1)}\bar{z}_n^0)' =$$

$$= s(p|f_1(z_0)|^{2(p-1)}\overline{f_1(z_0)}, \ldots, p|f_n(z_0)|^{2(p-1)}\overline{f_n(z_0)})'.$$

If we put $u = f(z_o)$ and $v = ([Df(z_0)]^{-1})'((p|z_1^0|^{2(p-1)}\bar{z}_1^0, \ldots, p|z_n^0|^{2(p-1)}\bar{z}_n^0)'$ where $z_0 = (z_1^0, \ldots, z_n^0)'$, then $(u, v) \in X_s(M)$ and from the definition of the class $Y(\Omega, M)$ we get that $\| h(u, v) \|_{2p} \geq M$, which is a contradiction with the hypothesis, and therefore $\| f(z) \|_{2p} < M$ in $B_{2p}$.

Another application of the THEOREM 3.2 is contained in the next result.

THEOREM 4.2. *Let $a$, $b$ two functions defined on the unit ball $B$ which satisfy*

$$|a(z)| - |b(z)| \geq 1, \tag{4.1}$$

*for all $z \in B$. Let $f \in H(B)$ be locally biholomorphic with $f(0) = 0$ and suppose that*

$$\| a(z)f(z) + b(z)([Df(z)]^{-1})'\bar{z} \| < 1 \tag{4.2}$$

*for all $z \in B$, then $\| f(z) \| < 1$ in $B$.*

*Proof.* Indeed, if the relation $\| f(z) \| < 1$ doesn't holds in all points of $B$, then, using the fact that $f(0) = 0$ and the continuity of the Euclidian norm, we deduce that there exists $z_0 \in B$ such that

$$1 = \| f(z_0) \| = \max\{\| f(z) \| : \| z \| = \| z_0 \|\}.$$

In view of COROLLARY 3.1, we can get a real positive number $s$ with $0 < s \leq \| z_0 \|^2$ and $([Df(z_0)]^{-1})'\bar{z}_0 = s\overline{f(z_0)}$. Using the condition (4.1) we conclude that,

$$\| a(z_0)f(z_0) + b(z_0)([Df(z_0)]^{-1})'\bar{z}_0 \| \geq 1,$$

which is a contradiction with (4.2), hence $\| f(z) \| < 1$, for all $z \in B$.

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