LIBERTAS MATHEMATICA, VOL. XVI (1996)

ON A NON CLASSICAL PERTURBED BOUNDARY OPTIMAL CONTROL SYSTEM

Taoufiq Benkiran

For a pretty little baby named Myriam

Abstract

This article deals (under some conditions) with the study of existence and behavior of the optimal control and the state of a non classical perturbed boundary optimal control linear system which arises in Aerodynamics.

0. Introduction and setting of the problem

Let Ω be a regular bounded open set and simply connected in the Euclidean space $\mathbf{R}^n$, with $\Gamma = \partial\Omega$ its boundary assumed to be smooth. Let U_{ad} be a closed vectorial subspace of $L^2(\Gamma)$. For $v \in L^2(\Gamma)$, we consider the perturbed problem:

$$(P_\epsilon)(v) \qquad \begin{aligned} & -\Delta y_\epsilon(v) = 0 \quad \text{in } \Omega, \\ & \tfrac{\partial}{\partial \nu} y_\epsilon(v) + \epsilon y_\epsilon(v) = v \text{ on } \Gamma, \\ & y_\epsilon(v) \in H^1(\Omega). \end{aligned}$$

Let's denote by T_ϵ the map $T_\epsilon: L^2(\Gamma) \to L^2(\Gamma)$

$$v \to T_\epsilon(v) = y_\epsilon(v) \mid_\Gamma,$$

where $y_\epsilon(v) \mid_\Gamma$ denotes the Γ-trace of the solution $y_\epsilon(v)$ of $(P_\epsilon)(v)$. Let us also consider the problem:

$$(M_\epsilon), \qquad J_\epsilon(u_\epsilon) = Min\{J_\epsilon(v); v \in U_{ad}\}, \text{ with } J_\epsilon(v) := \| T_\epsilon(v) - z \|^2$$

$\| . \|$ being the $L^2(\Gamma)$-norm, z a fixed element not vanishing of $L^2(\Gamma)$. We denote by

$$U_0 = \left\{ v \in L^2(\Gamma); \int_\Gamma v d\Gamma = 0 \right\} \text{ and } V = \left\{ y \in H^1(\Omega); \int_\Gamma y|_\Gamma d\Gamma = 0 \right\}$$

We are concerned with:

***Section 1:**

When U_{ad} has finite dimension, we prove:

(i) The existence and uniqueness of control u_ϵ in U_{ad}.

(ii) $(u_\epsilon, y_\epsilon(u_\epsilon))$ converges in $L^2(\Gamma) \times H^1(\Omega)$.

***Section 2:**

By an example, we prove:

If U_{ad} has infinite dimension, then the problem (M_ϵ) does not have a solution, in general.

1. Existence and Convergence of State and Control for the Systems $(P_\epsilon)(v)$ and (M_ϵ)

The space of admissible controls U_{ad} will be a linear closed subspace in $L^2(\Gamma)$ of finite dimension. Before beginning the study of the existence and convergence of state and control for the $(P_\epsilon)(v)$ and (M_ϵ) system, we prove the following lemma and make some remarks.

Lemma 1.1:

a) For $p \geq 0$, we consider the map: $\Psi_p : H^1(\Omega) \to \mathbf{R}^+$

$$y \to \Psi_p(y) = \sqrt{\| \nabla y \|^2_{L^2(\Omega)} + p\int_\Gamma | y_\Gamma |^2 d\Gamma} \quad .$$

(i) Ψ_0 is a norm in $V = \{y \in H^1(\Omega);\ \int_\Gamma y \mid_\Gamma d\Gamma = 0\}$ equivalent to the usual norm

$\| \cdot \|_{H^1(\Omega)}$ in $H^1(\Omega)$.

(ii) If $p > 0$, the map Ψ_p is a norm in $H^1(\Omega)$ equivalent to the usual norm $\| \cdot \|_{H^1(\Omega)}$. (i.e., there exists a constant A_p such that: $A_p \| y \|_{H^1(\Omega)} \leq \Psi_p(y)$).

b) Given $v \in L^2(\Gamma)$, if $y_\epsilon(v)$ is a solution of problem $(P_\epsilon)(v)$, then we have:

(i) If $v \in U_0$, $y_\epsilon(v)$ converges in $H^1(\Omega)$ to the solution $y_0(v)$ of the problem:

$$(Q_0)(v) \qquad \begin{aligned} & -\Delta y_0(v) = 0 \ \text{ in } \Omega \\ & \tfrac{\partial}{\partial \nu}\, y_0(v) = v \text{ on } \Gamma \\ & \int_\Gamma y_0(v) \mid_\Gamma d\Gamma = 0 \end{aligned}$$

(ii) If $v \notin U_0$, then $T_\epsilon(v)$ is not bounded in $L^2(\Gamma)$.

Proof:

a) (See [6]).

b) (i) If $\int_\Gamma v d\Gamma = 0$ $(v \in U_0)$, it is clear that $y_\epsilon(v) \in V = \{y \in H^1(\Omega); \int_\Gamma y \mid_\Gamma d\Gamma = 0\}$. Let us multiply the equation $(P_\epsilon)(v)$ by $y_\epsilon(v)$. After integrating and using the trace theorem (cf. [6]), we have:

$$A_0^2 \ \| T_\epsilon(v) \|^2 \leq A_0^2 \| y_\epsilon(v) \|^2_{H^1(\Omega)} \leq (\Psi_0(y_\epsilon(v)))^2 \leq \ \| v \| \ \ \| y_\epsilon(v) \mid_\Gamma \|$$

In particular, there exists a constant C independent of ϵ, such that $\| y_\epsilon(v) \|_{H^1(\Omega)} \leq C$.

Consequently, $y_\epsilon(v)$ converges in $H^1(\Omega)$ to the solution $y_0(v)$ of the problem $(Q_0)(v)$.

(ii) If $\int_\Gamma v d\Gamma \neq 0$ $(v \in U_0)$, we have: $0 = \int \Delta y_\epsilon(v) 1 dx = \int_\Gamma \frac{\partial}{\partial\nu} y_\epsilon(v) \mid_\Gamma d\Gamma$. Then:

$\int_\Gamma v d\Gamma = \epsilon \int_\Gamma y_\epsilon(v) \mid_\Gamma d\Gamma$, consequently: $\frac{1}{\epsilon} \mid \int_\Gamma v d\Gamma \mid \leq \sqrt{\mid \Gamma \mid} \parallel T_\epsilon(v) \parallel$ $(\mid \Gamma \mid = \int_\Gamma 1 d\Gamma)$.

<u>Remarks 1.1:</u>

1°) Assume that $\int_\Gamma v d\Gamma \neq 0$ $(v \notin U_0)$.

(i) For $\epsilon > 0$, the problem $(P_\epsilon(v))$ has a unique solution: Ψ_ϵ is a norm in $H^1(\Omega)$ equivalent to the usual norm $\parallel \cdot \parallel_{H^1(\Omega)}$.

(ii) For $\epsilon = 0$, the problem:

$$(P_0)(v) \qquad \begin{aligned} &-\Delta y(v) = 0 \quad \text{in } \Omega \\ &\frac{\partial}{\partial\nu} y(v) = v \quad \text{on } \Gamma \\ &y(v) \in H^1(\Omega) \end{aligned}$$

does not have a solution.

2°) If $\int_\Gamma v d\Gamma = 0$, the problem $(P_0)(v)$ has infinitely many solutions.

3°) the map T_ϵ is linear, continuous and one-to-one.

1.1 Existence and Convergence of Control for the $(P_\epsilon(v))$ and (M_ϵ) System

We begin by studying the following system (where $w \in U_0$ is given):

$$(H)(w) \qquad \begin{aligned} &-\Delta y(w) = 0 \quad \text{in } \Omega \\ &\frac{\partial}{\partial\nu} y(w) = w \quad \text{on } \Gamma \\ &y(w) \in H^1(\Omega) \end{aligned}$$

and

$$(N_0)(w) \qquad J_w(y) = Min\{J_w(s); s \in E(w)\} \text{ with } J_w(s) := \parallel s \mid_\Gamma - z \parallel^2$$

and where

$$E(w) = \{y \in H^1(\Omega);\ -\Delta y = 0 \text{ and } \frac{\partial}{\partial\nu} y \mid_\Gamma = w\}.$$

<u>Theorem 1.1:</u>

For each v in U_0, the system $(H)(v)$ or $(N_0)(v)$ has a unique solution $y(v)$ in $E(v)$ given by

$$(1.1) \qquad y(v) = y_0(v) + \frac{1}{|\Gamma|}\int_\Gamma z d\Gamma,$$

where $y_0(v)$ is the solution of the $(Q_0)(v)$ problem.

__Proof:__

*The map J_v is continuous and strictly convex.

Let $(s_1, s_2) \in (E(v))^2$ be such as: $J_v\left(\frac{s_1+s_2}{2}\right) = \frac{1}{2}(J_v(s_1) + J_v(s_2))$. Then after calculations,

we have: $\| (s_1 + s_2) \mid_\Gamma - 2z \|^2 = 2(\| s_1 \mid_\Gamma - z \|^2 + \| s_2 \mid_\Gamma - z \|^2)$; consequently

$\| s_1 \mid_\Gamma - s_2 \mid_\Gamma \|^2 = (\| s_1 \mid_\Gamma - z \|_2 + \| s_2 \mid_\Gamma - z \|_2 - 2 < s_1 \mid_\Gamma - z, s_2 \mid_\Gamma - z >) = 0$, which implies that $s_1 \mid_\Gamma = s_2 \mid_\Gamma$. So $\Delta s_1 = \Delta s_2$, and then $s_1 = s_2$.

*For each sequence (s_n) of $E(v)$ with $\lim_{n \to +\infty} \| s_n \|_{H^1(\Omega)} = +\infty$, $\lim_{n \to \infty} J_v(s_n) = +\infty$.

Let $s_n \in E(v)$, then: $\| \nabla s_n \|^2_{L^2(\Omega)} = \int_\Gamma v s_{n|\Gamma} d\Gamma$, so that from Lemma 1.1 (for $p = 1$) and by the Poincaré inequality, we have:

(1.2) $$A_{\frac{1}{2}} \| s_n \|^2_{H^1(\Omega)} \leq (\Psi_{\frac{1}{2}}(s_n))^2 \leq \| s_n \mid_\Gamma \|^2_{L^2(\Gamma)} + \frac{1}{2} \| v \|^2_{L^2(\Gamma)}$$

and $\lim_{n \to +\infty} J_v(s_n) = +\infty$.

Consequently (cf[4]), the system $(H)(v)$ or $(N_0)(v)$ has a unique solution $y(v)$ in $E(v)$.

*The $y_0(v)$ solution of $(Q_0)(v)$ problem is an element of $E(v)$, and any other element of $E(v)$ can be written as $s = y_0(v) + a$ (where a is a constant). For s in $E(v)$, let us write J_v in the following form:

(1.3)

$$J_v(s) = < (y_0(v) \mid_\Gamma - z) + a, (y_0(v) \mid_\Gamma - z) + a > = \| y_0(v) \mid_\Gamma - z \|^2 - 2a \int_\Gamma z d\Gamma + a^2 \mid \Gamma \mid.$$

J_v is optimum for $a = \frac{1}{|\Gamma|} \int_\Gamma z d\Gamma$, so that $y(v) = y_0(v) + \frac{1}{|\Gamma|} \int_\Gamma z d\Gamma$, and finally

(1.4) $$J_v(y(v)) = \| y_0(v) \mid_\Gamma - z \|^2 - \frac{1}{|\Gamma|} (\int_\Gamma z d\Gamma)^2$$

For the $(P_\epsilon)(v)$ and (M_ϵ) system, we have the following theorem:

__Theorem 1.2:__

(i) The solution u_ϵ of the problem (M_ϵ) converges in $L^2(\Gamma)$ to $u \in U_{ad} \bigcap U_0$.

(ii) The solution $y_\epsilon(u_\epsilon)$ of the problem $(P_\epsilon)(u_\epsilon)$ converges in $H^1(\Omega)$ to an element $s = y_0(u) + \beta$, with

(1.5) $$\beta \in [0, \frac{2}{|\Gamma|} \int_\Gamma z d\Gamma] \quad \text{or} \quad [\frac{2}{|\Gamma|} \int_\Gamma z d\Gamma, 0].$$

(iii) Moreover if $1_\Gamma \in U_{ad}$, $y_\epsilon(u_\epsilon)$ converges in $H_1(\Omega)$ to the solution $y(u)$ of the system $(H)(u)$ and $(N_0)(u)$ (where $1_\Gamma(x) = 1$ if $x \in \Gamma$ and 0 elsewhere).

Proof:

(i) The existence and uniqueness of u_ϵ is a consequence of the fact that the following two conditions are satisfied (cf [3]):

- The map: $v \to J_\epsilon(v)$ is l.s.c. (i.e., lower semi-continuous) on the space U_{ad} and strictly convex.
- For all sequences (v_n) of U_{ad} such that

$$\lim_{n \to +\infty} \| v_n \| = +\infty, \text{ then } \lim_{n \to +\infty} J_\epsilon(v_n) = +\infty.$$

U_{ad} has finite dimension, so that the set $G(T_\epsilon^{-1}) = \{(T_\epsilon v, v); v \in U_{ad}\}$ is closed in $L^2(\Gamma) \times L^2(\Gamma)$, consequently T_ϵ^{-1} is linear, continuous (cf.[2]) and so, there exists a constant $K > 0$ such that $\| v \| \leq K \| T_\epsilon(v) \|$ for each v in U_{ad}.

In particular, $\| v_n \| \leq K \| T_\epsilon(v_n) \|$, so if $\| v_n \| \to +\infty$ then $\lim_{n \to +\infty} J_\epsilon(v_n) = +\infty$.

Let $\{\phi_i \in J\}$ be an orthonormal basis of U_{ad} (where J is a finite subset of N). The control will be written as follows:

$$u_\epsilon = \sum_{i \in J} < u_\epsilon, \phi_i > \phi_i \tag{1.6}$$

where $<,>$ designates the scalar product in $L^2(\Gamma)$. Then:

$$T_\epsilon(u_\epsilon) = \sum_{i \in J} < u_\epsilon, \phi_i > T_\epsilon(\phi_i). \tag{1.7}$$

But u_ϵ is a solution of (M_ϵ), so: $J_\epsilon(u_\epsilon) \leq J_\epsilon(v)$ for all $v \in U_{ad}$. In particular $J_\epsilon(u_\epsilon) \leq J_\epsilon(0) = \| z \|^2$, consequently $T_\epsilon(u_\epsilon)$ is bounded on $L^2(\Gamma)$, so that for each $i \in J$, $< u_\epsilon, \phi_i >$ tends to a_i. Then u_ϵ converges in $L^2(\Gamma)$ to $u = \sum_{i \in J} a_i \phi_i$. On the other hand

$$\left| \int_\Gamma u_\epsilon \, d\Gamma \right| = \epsilon \left| \int_\Gamma y_\epsilon(u_\epsilon) \mid_\Gamma d\Gamma \right| \leq \epsilon \sqrt{|\Gamma|} \, \| T\epsilon(u_\epsilon) \|,$$

hence $u \in U_{ad} \bigcap U_0$.

(ii) Let us multiply the equation $(P_\epsilon)(u_\epsilon)$ by $y_\epsilon(u_\epsilon)$. After integrating, we have:

$$[\Psi_\epsilon(y_\epsilon(u_\epsilon))]^2 = \int_\Gamma T_\epsilon(u_\epsilon) u_\epsilon d\Gamma \leq \| T_\epsilon(u_\epsilon) \| \, \| u_\epsilon \|.$$

So $T_\epsilon(u_\epsilon)$ is bounded in $L^2(\Gamma)$, in particular $\| \nabla y_\epsilon(u_\epsilon) \|_{L^2(\Omega)}$ is bounded. Since Ψ_1 is a norm equivalent to the usual norm $\| . \|_{H^1(\Omega)}$, then there exists a constant C independent of ϵ such as: $\|y_\epsilon(u_\epsilon)\|_{H_1(\Omega)} \leq C$. Consequently $y_\epsilon(u_\epsilon)$ converges in $H^1(\Omega)$ to $s \in E(u)$:

$$\| \nabla(y_\epsilon(u_\epsilon) - s) \|^2_{L^2(\Omega)} = \int_\Gamma (y_\epsilon(u_\epsilon) - s) \mid_\Gamma (u_\epsilon - u - \epsilon y_\epsilon(u_\epsilon) \mid_\Gamma) d\Gamma \to 0.$$

Now $s \in E(u)$, so $s = y_0(u) + \beta$ and using the fact that u_ϵ is the solution of the (M_ϵ) problem, we have: $\| y_\epsilon(u_\epsilon) \mid_\Gamma - z \|^2 \leq \| y_\epsilon(u) \mid_\Gamma - z \|^2$, and passing to the limit as $\epsilon \to 0$, we obtain

(1.8) $$J_u(s) = \| y_0(u) \mid_\Gamma - z \|^2 - 2\beta \int_\Gamma z d\Gamma + \beta^2 \mid \Gamma \mid \leq J_u(y_0(u)) = \| y_0(u) \mid_\Gamma - z \|^2$$

and $\beta(\beta \mid \Gamma \mid - 2\int_\Gamma z d\Gamma) \leq 0$, which proves the assertion (1.5).

(iii) If $1_\Gamma \in U_{ad}$ then $k_\epsilon = u + \frac{\epsilon}{|\Gamma|}\int_\Gamma z d\Gamma \in U_{ad}$ and $y_\epsilon(k_\epsilon) = y_\epsilon(u) + \frac{1}{|\Gamma|}\int_\Gamma z d\Gamma$ is the solution of the $(P_\epsilon)(k_\epsilon)$ problem. $(y_\epsilon)(k_\epsilon)$ converges in $H^1(\Omega)$ to $Y(u) = y_0(u) + \frac{1}{|\Gamma|}\int_\Gamma z d\Gamma$. On the other hand: $\| y_\epsilon(u_\epsilon)_\Gamma - z \| \leq \| y_\epsilon(u)_\Gamma + \frac{1}{|\Gamma|}\int_\Gamma z d\Gamma - z \|$; then we have:

(1.9) $$J_u(s) = \| s|_\Gamma - z \|^2 \leq J_u(y(u)) = \| y(u)|_\Gamma - z \|^2$$

So, $y(u)$ is the solution of $(H)(u)$ and $(N_0)(u)$ system, and thus: $s = y(u)$ (i.e.: $\beta = \frac{1}{|\Gamma|}\int_\Gamma z d\Gamma$).

2. Example: If U_{ad} has infinite dimension, then the problem (M_ϵ)does not have a solution in U_{ad}, in general

Let $\Omega := B(0,1)$ be the unit ball in $\mathbb{R}^2$ and $\Gamma = S(0,1)$ it's boundary. We take

$U_{ad} = U = \left\{ v \in L^2(S(0,1)), \int_0^{2\pi} v(\xi) d\xi = 0 \right\}$, and we have:

Lemma 2.1:

(i) $\text{Min}\{J_\epsilon(v), v \in U\} = \| z - P_u(z) \|^2$.

where $P_u(z)$ is the orthogonal projection of z onto U.

(ii) If $z \notin H^1(\Gamma)$, then the problem (M_ϵ) does not have a solution in U.

Proof:

(i) For $z \in L^2(\Gamma)$, $J_\epsilon(v) := \| T_\epsilon(v) - P_u(z) \|^2 + \| P_u(z) - z \|^2$, there exists a sequence (z_j) of $H^1(\Gamma) \cap U$ such that $\lim_{j \to +\infty} \| z_j - P_u(z) \| = 0$. We consider the problem

$$(Q_j)\qquad \begin{aligned} &-\Delta\Psi_j = 0 \quad \text{in } \Omega,\\ &\Psi_j \mid_\Gamma = z_j \quad \text{on } \Gamma,\\ &\int_\Gamma \Psi_j \mid_\Gamma d\Gamma = 0. \end{aligned}$$

Let us write $v_{j,\epsilon} = \frac{\partial}{\partial\nu}\Psi_j \mid_\Gamma + \epsilon\Psi_j \mid_\Gamma$, so that clearly Ψ_j is also a solution of the problem $(P_\epsilon)(v_{j,\epsilon})$.

But $\lim\limits_{j\to+\infty} \| T_\epsilon(v_{j,\epsilon}) - P_u(z) \|^2 = \lim\limits_{j\to+\infty} \| \Psi_j\Gamma - P_u(z) \|^2 = \lim\limits_{j\to+\infty} \| z_j - P_u(z) \|^2 = 0.$

Consequently, $\lim\limits_{j\to+\infty} J_\epsilon(\Psi_j) = \| z - P_u(z) \|^2$. Finally, we obtain:

$$(2.1)\qquad Min\{J_\epsilon(v), v \in U\} = \| z - P_u(z) \|^2.$$

(ii) If $z \notin H^1(\Gamma)$, then $P_u(z) \notin H^1(\Gamma)$. Let us suppose the problem (M_ϵ) has a solution u_ϵ in U, then $T_\epsilon(u_\epsilon) = P_u(z)$, because $Min\{j_\epsilon(v), v \in U_{1d}\} = \| z - P_u(z) \|^2)$, (see (2.1)). This is a contradiction (since $P_u(z) \notin H^1(\Gamma)$ and $T_\epsilon(u_\epsilon) \in H^1(\Gamma)$).

References

1. **T. Benkiran,** Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal. Thèse de docteur, Faculté des Sciences Semlalia, Univ. Cadi Ayyad, Marrakech (15 Juin 1995).
2. **H. Brezis,** Analyse fontionnelle, Théorie et applications, Masson, Paris 1983.
3. **D. Huet,** Décomposition Spectrale et Opérateurs. (Presses Universitaires de France, Paris, 1977).
4. **J.L. Lions,** Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles, Dunod Gauthier-Villars, Paris 1968.
5. **J.L. Lions,** Perturbation Singulière dans les problèmes aux limites et en control optimal, Springer-Verlag, Berlin 1973.
6. **J.L. Lions and E. Magenes,** Problèmes aux limites nonhomogènes et applications, Vol. 1, Dunod, Paris 1968.