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# Asymptotic Weak Almost Periodicity and Almost-Orbit of Semigroups of Operators

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**Abstract**

We present some results of weakly almost periodic functions with values in a Banach space. Then we apply these results to investigate the weakly almost periodic solutions of some differential equations in a Banach space via a technic of semigroups operators. This work is motivated by Some papers of Ruess, Summers and Phong.

**Key words:** Asymptotic weakly almost periodic- Semigroup operator- Almost-orbit-carleman spectrum- compact- resolvant - Fourier transform.

AMS Subject Classification: 34K20 34K25

## 1 Introduction

The point of departure for this work is the problem of studying under which conditions the abstract cauchy problem :

$$\frac{dx}{dt}(t) = Ax(t) + f(t),$$

where $A$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of bounded linear operators in a Banach space $X$ and $f$ is a given $X$-valued fonction on $\mathbb{R}$, has solutions which are weakly almost periodic,.i.e. those for

which their set of translates is weakly relatively compact in the sup-normed space $BC((J,X),\|.\|_\infty)$ (for $J \in \{I\!R,\ I\!R^+\}$) of all bounded continuous functions from $J$ into $X$. In comparison with the existing literature of this problem ([4], [6]) with the results based on the assumption that $\sigma(A) \cap iI\!R$ is empty or countable, we investigate the situation where $A$ is the generator of a uniformly bounded $C^0$−semigroup, and $\sigma(A) \cap iI\!R$ not necessary empty nor countable.
The paper is organised as, the first section is devoted with some preliminary results on weak almost periodic functions, section (2), deals with some applications to some abstract differentiel equation.

## 1.1 Preliminaries.

Throughout the paper, $X$ will denote a ( real or complex ) Banach space, which we shall tacitly assume to be complex whenever spectral properties of linear operators enter the picture. For $J \in \{I\!R, I\!R^+\}$, the sup-normed space of all bounded continuous and bounded uniformly continuous functions defined on $J$ with values in $X$, will be denoted by $BC(J,X)$ and $BUC(J,X)$ respectively. Furthermore, $(T(t))_{t\geq 0}$ will here after denote a $C^0$−semigroup of bounded linear operators on $X$. For $u \in BC(J,X)$, and $w \in J$, moreover, we put $u_w(t) = u(t+w)$, $t \in J$.

The results of this paper make essential use of the notion of weakly almost periodic and asymptotically almost periodic functions as introduced in the scalar case by Eberlein ([5]). A functin $f \in BC(J,X)$, is said to be weakly almost periodic (in the sense of Eberlein) if the orbit of $f$ with respect to $J$

$$O_J \;=\; \{f_t : t \in J\},$$

is relatively compact with respect to the weak topology of $BC(J,X)$. The space of all such functions will be denoted by $W(J,X)$. $W_0(J,X)$ denotes the closed subspace of all $f \in W(J,X)$ such that some sequence $(f_{t_n})_n$ of translates of $f$ is weakly convergent to the zero function $f \in BUC(I\!R,X)$ is called weakly asymptotically almost periodic (in the sense of Eberlein) if the set

$$\{(f/_{R^+})_t,\ t \geq 0\}$$

of translates of its restrection to $I\!R^+$ is weakly relatively compact in $BUC(I\!R^+,X)$ (endowed with the sup-norm). The space of all such functions will be denoted by $W^+(I\!R,X)$.

Results of deleeuw and Glicksberg imply that the following decompositions

$$W(J,X) = AP(\mathbb{R},X)/_J \oplus^t W_0(J,X)$$

$$W^+(\mathbb{R},X) = AP(\mathbb{R},X) \oplus^t W_0^+(J,X)$$

hold. Here $AP(\mathbb{R},X)$ denotes the space of almost periodic functions,i.e.

$$AP(\mathbb{R},X) := \left\{ \begin{array}{c} f \in BC(\mathbb{R},X) : O_R(f) \\ \text{is relatively compact in } (BC(\mathbb{R},X), \|.\|_\infty) \end{array} \right\},$$

and $W_0^+(J,X)$ denotes the subspace of $W^+(\mathbb{R},X)$ defined as follow :

$$W_0^+(J,X) := \{f \in BUC(\mathbb{R},X) : f/_{\mathbb{R}^+} \in W_0(\mathbb{R}^+,X)\}.$$

**Theorem 1** *[7] Let $u \in BC(\mathbb{R}^+,X)$. The following results are equivalent :*
*(i) $u \in W(\mathbb{R}^+,X)$;*
*(ii) given any sequences $((t_m,x_m^*))_m$ in $\mathbb{R}^+ \times B_{X^*}$ and $(w_n)_n$ in $\mathbb{R}^+$,*

$$\lim_{n\to\infty}\lim_{m\to\infty} \langle u(t_m+w_n), x_m^*\rangle = \lim_{m\to\infty}\lim_{n\to\infty} \langle u(t_m+w_n), x_m^*\rangle,$$

*when both iterated limits exist.*

The equivalence of $(i)$ and $(ii)$ is established in [3] and [8].
For a function $u : \mathbb{R}^+ \to X$, we will refer to

$$\gamma(u) := \{u(t),\ t \in \mathbb{R}\}$$

as the orbit of $u$.

**Theorem 2** *[7] Let $u \in BC(\mathbb{R}^+,X)$, and consider the following assertions :*
*(i) $u \in W(\mathbb{R}^+,X)$;*
*(ii) $\|.\| - \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)\,dt$ exists;*
*(iii) $\gamma(u)$ is weakly relatively compact in $X$.*
*Then $(i)$ implies both $(ii)$ and $(iii)$.*

The fact that $(i)$ implies $(ii)$ is proved by Eberlien in ([5]). The others implications are obvious.

### 1.1.1 A spectral Creterion for almost periodicity and asymptotic weakly almost periodicity.

**Definition 3** *For $u \in BUC(\mathbb{R}, X)$, the set of regular points $R(u)$ of $u$ is defined as the set of points $\lambda \in \mathbb{R}$ which have a neighborhood $\vartheta$ such that $\Psi * u \equiv 0$ whenever $\Psi \in L^1(\mathbb{R})$ and $supp\widehat{\Psi} \subset \vartheta$, ($\widehat{\Psi}$ denotes the fourier transform of $f$).*

**Definition 4** *For $u \in BUC(\mathbb{R}, X)$, the spectrum $sp(u)$ is defined as the complement in $\mathbb{R}$ of $R(u)$.*

**Definition 5** *For $u \in BUC(\mathbb{R}, X)$, the almost periodic spectrum $\Delta_{AP}(u)$ of $u$ (resp. the weakly asymptotical almost periodic spectrum $\Delta_{W^+}(u)$ of $u$ is defined as the complement in $\mathbb{R}$ of points $\lambda \in \mathbb{R}$ which have a neighborhood $\vartheta$ such that $\Psi * u \in AP(\mathbb{R}, X)$ (resp $\Psi * u \in W^+(\mathbb{R}, X)$) whenever $\Psi \in L^1(\mathbb{R})$ and $supp\widehat{\Psi} \subset \vartheta$. ( where $\widehat{\Psi}$ designes the Fourier transform of $\Psi$).*

From the definition, we have the following obvious inclusions :

$$\begin{aligned} \Delta_{AP}(u) &\subset sp(u) \\ \Delta_{W^+}(u) &\subset sp(u). \end{aligned}$$

**Proposition 6** *[6] Let $u \in BUC(\mathbb{R}, X)$. Then :*
*i) $sp(u) = \emptyset$ if and only if $u \equiv 0$.*
*ii) $\Delta_{AP}(u) = \emptyset$ (resp $\Delta_{W+}(u) = \emptyset$) if and only if $u \in AP(\mathbb{R}, X)$ (resp $u \in W^+(\mathbb{R}, X)$.*

**Proposition 7** *Let $u$, $v \in BUC(\mathbb{R}, X)$. Then :*
*i) if $u$ is derivable such that $\frac{du}{dt} \in BUC(\mathbb{R}, X)$, then $sp(\frac{du}{dt}) \subset sp(u)$;*
*ii) $sp(u + v) \subset sp(u) \cup sp(v)$.*

### 1.1.2 Characterization of almost periodic and asymptotically weakly almost periodic functions by their spectrums.

Let $X_0$ denotes the vector-space defined by :

$$C_0 := \left\{ (x_n)_n \subset \mathbb{R}, \lim_{n \to \infty} x_n = 0 \right\}.$$

**Theorem 8** *[6] Assume that $u \in BUC(\mathbb{R}, X)$, and $\Delta_{AP}(u)$ is countable. Then $u$ is almost periodic provided one of the following conditions holds :*
*i) $u$ has a weakly relatively compact range;*
*ii) $X$ does not contain an isomorphic copy of $C_0$;*
*iii) for every $\lambda \in \Delta_{AP}(u)$, the function $\exp(-i\lambda t)\, u(t)$ has uniformly convergent mean,.i.e.*

$$\|.\| - \lim_{T\to\infty} \frac{1}{T}\int_{\alpha}^{\alpha+T} \exp(-i\lambda t)\, u(t)dt \;=\; x \in X$$

*exists uniformly for all $\alpha \geq 0$..*

**Theorem 9** *[6] Assume that $u \in BUC(\mathbb{R}, X)$, $\Delta_{W^+}(u)$ is countable, and for every $\lambda \in \Delta_{W^+}(u)$, $\exp(-i\lambda t)\, u(t)$ has uniformly convergent mean. Then $u$ is weakly asymptotically almost periodic.*

### 1.1.3 Asymptotically almost periodicity for almost-orbit of uniformly bounded $C^0$−semigroup.

**Definition 10** *A continuous function*

$$u \;: \mathbb{R}^+ \;\to\; X,$$

*is said to be almost-orbit of $(T(t))_{t\geq 0}$ if*

$$\lim_{t\to\infty} \sup_{h\geq 0} \|u(t+h) - T(h)u(t)\| \;=\; 0.$$

Assume that $(T(t))_{t\geq 0}$ is a uniformly bounded $C^0$−semigroup. If $f \in L^1(\mathbb{R}^+, X)$, then it readily follows that the function :

$$u(t) \;=\; T(t)x + \int_0^t T(t-s)f(s)\, ds \;,\; t \geq 0,$$

is an almost-orbit of $(T(t))_{t\geq 0}$.

**Theorem 11** *[7] Assume that $(T(t))_{t\geq 0}$ is a uniformly bounded $C^0$−semigroup, and let $u \;: \mathbb{R}^+ \;\to\; X$, be an almost-orbit of $(T(t))_{t\geq 0}$. Then $u$ is weakly almost periodic if and only if $\gamma(u)$ is weakly relatively compact in $X$.*

# 2 Differential equations.

In this section, we apply the above results to study under which conditions the solution of differential equation :

$$\frac{dx}{dt}(t) = Ax(t) + f(t), \qquad (1)$$

is weakly almost periodic, where $A$ is the generator of a uniformly bounded $C^0$−semigroup, and $f$ is a weakly asymptotically almost periodic function.

In the following, given $f \in BUC(\mathbb{R}, X)$, we shall denote by $M(f)$ the closed linear span in $BUC(\mathbb{R}, X)$ of $H(f)$,( where $H(f)$ denotes the hull of f see [5]) by $(S_f(t))_{t\in R}$ the restriction to $M(f)$ of the translation group $(S(t))_{t\in R}$on $BUC(\mathbb{R}, X)$, and by $D_f$ the generator of $(S_f(t))_{t\in R}$. We have the following result :

**Proposition 12** *[4] Let $f \in BUC(\mathbb{R}, X)$, then :*
*i) $i\ sp(f) = \sigma(D_f)$;*
*ii) if $sp(f)$ is compact, then $D_f$ is bounded.*

**Assuming that** :
$(H_1)$ $A$ is the generator of a bounded $C^0$−semigroup $(T(t))_{t\geq 0}$.
$(H_2)$ $f \in BUC^1(\mathbb{R}, X)$ is such that $sp(f)$ is compact and $i\ sp(f) \cap \sigma(A) = \emptyset$.

**Theorem 13** *Under the assumptions $(H_1)$ and $(H_2)$ if $f$ is weakly asymptotically almost periodic. Then equation (1) has one and only one bounded solution defined on $\mathbb{R}^+$ which is weakly asymptotically almost periodic.*

In order to prove this theorem we need the following lemma :

**Lemma 14** *[6] Let $X$ and $Y$ be two Banach spaces, and assume that*

$$A\ : D(A) \subset X \ \rightarrow\ X\ ,\quad C\ : Y\ \rightarrow\ X,$$

*and $D\ : Y \ \rightarrow\ Y$ are linear operators such that $A$ is closed, $C$ and $D$ are bounded, and $\sigma(A) \cap \sigma(D) = \emptyset$.*
*With $\rho(A)$ denoting the resolvent set of $A$, let $\Gamma$ be a cycle in $\rho(A)/\sigma(D)$ that surrounds $\sigma(D)$ in $\rho(A)$.*
*Then the operator :*

$$L\ := \frac{-1}{2\pi i}\int_\Gamma (A-\lambda)^{-1}C(D-\lambda)^{-1}d\lambda,$$

*is a bounded linear operator from $Y$ into $X$ such that :*
*i)* $Łu \in D(A)$, *for every* $u \in Y$;
*ii)* $AŁu - ŁDu = Cu$, *for every* $u \in Y$.

**Proof of theorem** : Since $sp(f)$ is a compact set, then $D_f$ is a bounded linear operator on $M(f)$.
$A$ is closed, because $A$ is the generator of a $\mathcal{C}^0-$semigroup. If we consider $C = \delta_0$, which is defined on $M(f)$ with values on $X$ by : $\delta_0(h) = h(0)$, for every $h \in M(f)$.
One has :

$$i\ sp(f) = \sigma(D_f),$$

and by hypothesis one has

$$\sigma(A) \cap i\ sp(f) = \emptyset,$$

hence forth

$$\sigma(A) \cap \sigma(D_f) = \emptyset.$$

By lemma 14,

$$Ł \ := \ \frac{-1}{2\pi i}\int_{\Gamma}(A-\lambda)^{-1}\delta_0(D_f-\lambda)^{-1}d\lambda,$$

is a bounded linear operator defined on $M(f)$ with values in $X$, and such that :

$$AŁh - ŁD_f h = \delta_0 h, \text{ for every } h \in M(f). \tag{2}$$

If we put :

$$u_f(t) \ := \ -Łf_t \quad \text{for } t \geq 0,$$

we have

$$\begin{aligned} \frac{du_f}{dt}(t) &= -Ł\frac{d}{dt}S_f(t)f \\ &= -ŁD_f S_f(t)f \\ &= -ŁD_f f_t. \end{aligned}$$

So by equality (2) we found that :

$$\begin{aligned} -ŁD_f f_t &= \delta_0 f_t - AŁf_t \\ &= f(t) + Au_f(t), \end{aligned}$$

so

$$\frac{du_f}{dt}(t) \;=\; Au_f(t) + f(t).$$

Henceforth, $u_f$ is a bounded solution of equation (1).
One has

$$\gamma(u_f) \;=\; \{u_f(t),\ t \in I\!R^+\}$$

$$= \; -\text{Ł}\{f_t,\ t \in I\!R^+\},$$

then, $\gamma(u_f)$ is relatively weakly compact.
$f \in BUC^1(I\!R, X)$. Then by proposition (3) in [1],

$$u_f(t) \;=\; T(t)u_f(0) + \int_0^t T(t-s)f(s)\,ds.$$

So $u_f$ is an almost-orbit of the uniformly bounded $C^0-$semigroup $(T(t))_{t\in R^+}$ with a relatively compact range, then, by theorem (11), $u_f$ is Asymptotically weakly almost periodic.

**Example 15** *: Let $X$ be a reflexif Banach space and $f$ the $X$-valued function defined on $I\!R$ by :*

$$f(t) \;=\; \sum_{k=1}^{n} \exp(-\lambda_k t)x_k,\ \textit{for } t \in I\!R,$$

*where $x_k \in X$, for $k \in \{1, 2, ....n\}$ , and $\lambda_k > 0$, for every $k \in \{1, 2, ....n\}$ are such that*

$$\sum_{k=1}^{n} \lambda_k x_k \;=\; 0. \tag{3}$$

*With (3), we guarantee that $f \in BUC(I\!R, X)$ is derivable and $\frac{df}{dt} \in BUC(I\!R, X)$. We have that*

$$f(t) \;=\; \sum_{k=1}^{n} f_k(t),$$

*where*

$$f_k(t) \;=\; \exp(-\lambda_k t)x_k.$$

*Let $f_k/_{R^+}$ denotes the restriction of $f_k$ on $I\!R^+$, then $f_k/_{R^+} \in L^1(I\!R^+, X)$ for every $k \in \{1, 2, ....n\}$ , so $f/_{R^+} \in L^1(I\!R^+, X)$.*

*For $a > 0$, let $(T_a(t))_{t\geq 0}$ denotes the bounded $C^0-$semigroup on $X$ defined as follows*

$$T_a(t)x = \exp(-at)x, \text{ for every } x \in X,$$

*$(T_a(t))_{t\geq 0}$ is a bounded $C^0-$semigroup on $X$, then by result of Summers ([7]), it follows that the function $f_a^x$ defined on $\mathbb{R}^+$ by :*

$$f_a^x(t) = \exp(-at)x,$$

*is weakly almost periodic function. Since*

$$f/_{R^+}(t) = \sum_{k=1}^{n} f_{\lambda_k}^{a_k}(t),$$

*then $f \in W^+(\mathbb{R}, X)$.*
*By proposition (7), we have that*

$$sp(f) \subset \bigcup_{k=1}^{n} sp(f_k).$$

*Since $sp(f_k) = \{\lambda_k, -\lambda_k\}$, then $sp(f)$ is compact.*
*Now we concider the following equation :*

$$\left\{\begin{array}{lcl} \dfrac{du}{dt} & = & Au(t) + \sum_{k=1}^{n} \exp(-\lambda_k t)x_k \\ u(0) & = & u_0 \in D(A); \end{array}\right. \tag{4}$$

*where $A$ is the generator infinitesimal of a stable $C^0-$semigroup $(T(t))_{t\geq 0}$ on $X$ (i.e. $\exists c > 0$ and $M \in \mathbb{R}^+$ such that $\|T(t)\| \leq M\exp(-ct)$, for all $t \geq 0$.), $x_k \in X$, for $k \in \{1, 2, ....n\}$, and $\lambda_k > 0$, for every $k \in \{1, 2, ....n\}$ such that*

$$\sum_{k=1}^{n} \lambda_k x_k = 0.$$

*Since $A$ is the infinitesimal generator of a stable $C^0-$semigroup, then*

$$\sigma(A) \cap i\mathbb{R} = \emptyset,$$

*in particular*

$$\sigma(A) \cap isp(f) = \emptyset,$$

*then, by Theorem 13 equation (4) has one solution defined on $\mathbb{R}^+$ wich is weakly almost periodic.*

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