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APPLICATION OF THE HAMILTON-JACOBI THEORY IN OPTIMAL NONSINGULAR CONTROL

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Abstract

This study presents the application of the canonic transformations theory in determining the optimal transfer problems' extremals. So we obtain a close form solution for the curve's arc problem by estimation of an undefined integral representing the complete solution. The use of a sufficient canonicity condition allows to define an advantageous canonical transformation and involves the determination of the generating function as a solution of the HamiltonJacobi equation.

1 Introduction.

Extremals representing the optimal orbital transfer trajectories are solutions of a Hamiltonian system. A canonical transformation will be necessary to provide a reduction of the initial canonical system. Since the integration constants are canonical constants, these might be used to define a basic solution regarding the canonical approximation of the perturbation in the optimal transfer problem for a low thrust space vehicle.

The curve's arc problem was solved by Eckenwiler [3], Tapley and Powers [4], have shown how the solution can be obtained by Hamiltonian methods. The solutions provided by the enlisted papers require many integrations and make the circular conditions appear as a special case and not as an extension of the elliptic case.

An efficient method for the elimination of the circular singularity having the canonical constants as integration constants is given by [7]. Using the Hamiltonian formulation method, this paper is providing the solution for the non-singular arc of trajectory.

2 Optimal transfer

We are considering the autonomous controlled system:

$$\dot{x}_i = f_i(x, u) \qquad (i = 1, \ldots, n) \tag{1}$$

where x is the n - dimension state vector and u the m - dimension control vector.

It is assumed that the functions $f_i(x,u)$, $\dfrac{\partial f_i(x,u)}{\partial x_j}$, $i,j = 1,\ldots,n$ are defined and continuous on the manifold $\mathbb{R}_n \times U$, so that for the initial conditions $x(t_0) = x^0$ and for a given control $u = u(t)$, the system (1) admits a unique solution.

We consider the performance index

$$J = \int_{t_0}^{t_f} f_0(x,u)dt \tag{2}$$

$$\dot{x}_0 = f_0(x,u) \tag{3}$$

so that the system to be analysed shall become:

$$\dot{x}_i = f_i(x,u) \quad (i = 0,\ldots,n) \tag{4}$$

We will consider the associate system:

$$\dot{\lambda}_i = -\sum_{j=0}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \quad (i = 0,\ldots,n) \tag{5}$$

which is defining the vectorial function $\lambda = (\lambda_0(t), \lambda_1(t), \ldots, \lambda_n(t))$ for the admissible control $u(t)$, corresponding to the trajectory $x(t)$. If the hamiltonian function is introduced:

$$H(x,u,\lambda) = \sum_{i=0}^{n} \lambda_i f_i(x,u) \tag{6}$$

equations (4) and (5) will be written in the canonic form:

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial \lambda_i} \\ \dot{\lambda}_i &= -\frac{\partial H}{\partial x_i} \quad (i = 0,\ldots,n) \end{aligned} \tag{7}$$

For $x(t)$, $\lambda(t)$ solution of system (7), there exists a control $u(t)$ for which the necessary optimum condition represented by Pontriagyn's Maximum Principle is satisfied:

$$H(x(t),u(t),\lambda(t)) = \sup_{u \in U} H(x(t),u,\lambda(t)) \tag{8}$$

For non - autonomous systems

$$\dot{x}_i = f_i(x,u,t) \quad (i = 1,\ldots,n) \tag{9}$$

and the performance index

$$J = \int_0^{t_1} f_0(x,u,t)dt \tag{10}$$

the assumptions made in the autonomous case are maintained on the manifold $\mathbb{R}_{n+1} \times U$.

With the substitution $t = x_{n+1}$, this case will be reduced to the case of autonomous systems. Thus we have:

$$\begin{aligned} \dot{x}_i &= f_i(x, u, x_{n+1}) \\ \dot{x}_{n+1} &= 1 \end{aligned} \tag{11}$$

and the varieties $S_0(t)$ and $S_1(t)$ become fixed varieties in the $n+1$ dimensions space $\overline{x} = (x_1, ..., x_n; x_{n+1})$.

By taking (11) into account the Hamiltonian will become

$$H^*(\overline{x}, u, \overline{\lambda}) = H(\overline{x}, u, \lambda) + \lambda_{n+1} \tag{12}$$

where

$$H(\overline{x}, u, \lambda) = \sum_{j=0}^{n} \lambda_j f_j(\overline{x}, u) \tag{13}$$

Use of the Maximum Principle determines the optimal control so that relation (11) will become:

$$H^*\left(\overline{x}, u, \overline{\lambda}\right) = H^*\left(\overline{x}, \overline{\lambda}\right) \tag{14}$$

Because $H^*(\overline{x}, \overline{\lambda})$ does not depend upon the independent variable, it follows that it is a prime integral of the canonic system, i.e.,

$$H^*\left(\overline{x}, \overline{\lambda}\right) = h \tag{15}$$

Let τ be the new independent variable defined by:

$$\frac{dt}{d\tau} = \phi\left(\overline{x}, \overline{\lambda}\right) \tag{16}$$

Considering the hamiltonian

$$\overline{H} = \phi(\overline{x}, \overline{\lambda})\left[H^* - h\right] \tag{17}$$

it will be shown that it verifies a canonic system

We get

$$\begin{aligned} \frac{\partial \overline{H}}{\partial \overline{x}} &= \frac{\partial \phi}{\partial \overline{x}}\left[H^*\left(\overline{x}, \overline{\lambda}\right) - h\right] + \phi \frac{\partial H^*}{\partial \overline{x}} = \frac{dt}{d\tau}\left(-\frac{d\overline{\lambda}}{dt}\right) = -\frac{d\overline{\lambda}}{d\tau} \\ \frac{\partial \overline{H}}{\partial \overline{\lambda}} &= \frac{\partial \phi}{\partial \overline{\lambda}}\left[H^*\left(\overline{x}, \overline{\lambda}\right) - h\right] + \phi \frac{\partial H^*}{\partial \overline{\lambda}} = \frac{dt}{d\tau} \cdot \frac{d\overline{x}}{dt} = \frac{d\overline{x}}{d\tau} \end{aligned} \tag{18}$$

For the optimal transfer we have to determine those trajectory arcs $\overline{x}$ corresponding to the optimal controls $\overline{u}$, which minimise the considered performance index.

3 Canonic transformations

Consider the canonic co-ordinate space, the Gibbs' space $(p, q) = (x, \lambda)$. A co-ordinate transformation in this space:

$$\begin{aligned} X_i &= X_i(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n, t) \\ \Lambda_i &= \Lambda_i(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n, t) \qquad (i = 1, \ldots, n) \end{aligned} \tag{19}$$

where X_i and Λ_i are C_2 class functions, so that:

$$\frac{\partial(X, \Lambda)}{\partial(x, \lambda)} \neq 0 \tag{20}$$

will be called canonic if it converts whatever canonic system

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, \ldots, n) \tag{21}$$

into a canonic system:

$$\dot{X}_i = \frac{\partial K}{\partial \Lambda_i} \qquad \dot{\Lambda}_i = -\frac{\partial K}{\partial X_i} \qquad (i = 1, \ldots, n) \tag{22}$$

The implicit functions theorem and the condition (20) provide that together with (19) the reversed transformation also exists and x_i, λ_i are C_2 class functions.

We consider the general form:

$$\omega = \sum_{i=0}^{n} \lambda_i dx_i - H dt \tag{23}$$

The transformation $\{X(x, \lambda, t), \Lambda(x, \lambda, t)\}$ is canonic if there exists a function $K(X, \Lambda, t)$ so that if we denote:

$$\Omega = \sum_{i=0}^{n} \Lambda_i dX_i - K dt \tag{24}$$

we shall have:

$$d\omega = d\Omega \tag{25}$$

Condition (25) is equivalent with the condition for the existence of a function $S(x, \lambda, X, \Lambda, t)$ so that

$$\omega = \Omega + dS \tag{26}$$

Taking into account (23) and (24), the equation (26) will become:

$$\sum_{i=0}^{n}(\lambda_i dx_i - \Lambda_i dX_i) + (K - H)dt = dS \tag{27}$$

For x_i, X_i, t the variation coincides with the differential.

As previously shown, along the optimal trajectory $dS = 0$.

The performance index of the optimum problem stated is invariant w. r. to the canonic transformation performed so that if the transformation is time - independent, it follows:

$$\sum_{i=1}^{n}(\lambda_i \delta x_i - \Lambda_i \delta X_i) = 0 \tag{28}$$

We consider the transformation:

$$x_i = \Phi(X) \tag{29}$$

By working out relation (28) it follows:

$$\Lambda_i = \sum_{j=1}^{n}\lambda_j \frac{\partial \Phi_j}{\partial X_i} \quad (i = 1, \ldots, n) \tag{30}$$

Let the motion equations of a space vehicle be linear w. r. to the thrust vector T.

The performed transformation will preserve the thrust's linearity in the new equations of motion.

When the thrust is 0, the space vehicle motion represents the natural trajectory for which $X_\alpha = c_\alpha \quad (\alpha = 1, \ldots, p)$ are prime integrals of the motion.

In this case the Hamiltonian may be written:

$$K_0 = \dot{X}_0 + \sum_{i=p+1}^{n}\Lambda_i X_i \tag{31}$$

We denote by K_T the term of the expression of K containing the thrust, thus we have:

$$K = K_0 + K_T \tag{32}$$

Taking into account the structure of the Hamiltonian, obtained by the change of the independent variable expressed by relation (17) it will result:

$$\overline{K} = \frac{dt}{dX_r}[K - C] \tag{33}$$

or for the analysed case

$$\overline{K}_0 = \frac{dt}{dX_r}[K_0 - C_0] \tag{34}$$

where $C_0 = K_0$ represents the value at the initial moment of the null thrust optimal trajectory arc.

4 Construction of the generating function

We consider the generating function

$$S(z,\lambda,X,\Lambda) \quad = \quad S_1(z_\alpha, X_\alpha) + S_2(z_k, \Lambda_k) \tag{35}$$

where the index α marks cyclic variables.

Denoting:

$$S_2(z_k,\Lambda_k) + \sum_k (\Lambda_k X_k)_{X_k \to (z_k,\Lambda_k)} \quad = \quad \overline{S}_2(z_k,\Lambda_k)$$

the sufficient canonicity condition may be written:

$$\sum_\alpha (\lambda_\alpha dz_\alpha - \Lambda_\alpha dX_\alpha) + \sum_k (\lambda_k dz_k + X_k d\Lambda_k) = \tag{36}$$
$$= \sum_\alpha \left(\frac{\partial S_1}{\partial z_\alpha} dz_\alpha + \frac{\partial S_1}{\partial X_\alpha} dX_\alpha \right) + \sum_k \left(\frac{\partial \overline{S}_2}{\partial z_k} dz_k + \frac{\partial \overline{S}_2}{\partial \Lambda_k} d\Lambda_k \right)$$

from which

$$\lambda_\alpha = \frac{\partial S_1}{\partial z_\alpha} \qquad \Lambda_\alpha = -\frac{\partial S_1}{\partial X_\alpha}$$
$$\lambda_k = \frac{\partial \overline{S}_2}{\partial z_k} \qquad X_k = \frac{\partial \overline{S}_2}{\partial \Lambda_k} \tag{37}$$

Consider the generating function (35) given by:

$$S(z,\lambda,X,\Lambda) \quad = \quad \sum_\alpha z_\alpha X_\alpha + \sum_k z_k \Lambda_k \tag{38}$$

Thus the system (37) is defining the transformation:

$$\lambda_\alpha = X_\alpha \qquad \lambda_k = \Lambda_k$$
$$z_\alpha = -\Lambda_\alpha \qquad z_k = X_k \tag{39}$$

From (39) it follows:

$$\lambda = \frac{\partial S}{\partial z} \tag{40}$$

Let X_r be the new independent variable. Determination of the generating function S which makes the transformed Hamiltonian vanish requires solving of the Hamilton - Jacobi equation:

$$\frac{\partial S}{\partial X_r} + \overline{K}_0 \left(z, \frac{\partial S}{\partial z}, X_r \right) \quad = \quad 0 \tag{41}$$

Finding the complete integral of the non - linear equation is equivalent with the integration of the transformed canonic system. Taking the time variable $t = x_{n+1}$ as state variable on the canonic system solution we have:

$$\overline{K}_0 = const = k \tag{42}$$

From (41) we obtain:

$$\frac{\partial S}{\partial X_r} = -\overline{K}_0 = -k \tag{43}$$

By integration equation (43) becomes

$$S = -kX_r + W(x_1, \ldots, x_n) \tag{44}$$

where W is the complete integral of the equation

$$\overline{K}_0\left(x_1, \ldots, x_n, \frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n}\right) = 0 \tag{45}$$

The unknown function W will be determined by using the variable separation method

$$W = \sum_i W_i(x_i) \tag{46}$$

Taking (40) and (44) into account we have

$$\lambda_i = \frac{\partial S}{\partial x_i} = \frac{\partial W}{\partial x_i} \quad (i = 1, \ldots, n) \tag{47}$$

From the prime integral of the canonic system one can get:

$$\lambda_i = f_i(x_i, a_1, \ldots, a_n, k) \tag{48}$$

so that from (47) it follows

$$W = \sum_i \int f_i(x_i, a_1, \ldots, a_n, k)\, dx_i \tag{49}$$

The complete integral of equation (41) will thus be:

$$S = -kX_r + \sum_a a_a x_a + \sum_k f_i(x_i, a_1, \ldots, a_n, k)\, dx_k \tag{50}$$

As stated by the Hamilton - Jacobi theorem the functions $x_i(t)$, $\lambda_i(t)$ determined by the system

$$\begin{aligned} \frac{\partial S}{\partial a_j} &= b_j = const \\ \frac{\partial S}{\partial x_j} &= \lambda_j \quad (j = 1, \ldots, n) \end{aligned} \tag{51}$$

are solutions of the canonic system.

5 Conclusions

The performed study concerns the use of canonical transformations in the study of optimal transfer. The generation function was determined as a complete integral of the Hamilton - Jacobi equation. The parametric form of the optimal transfer trajectories is obtained as a solution of the Hamilton - Jacobi system. Small variations of those control parameters are a characteristic of low thrust space vehicle evolutions.

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