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AN EXISTENCE RESULT FOR OSCILATORY STOKES FLOWS

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Abstract

In this paper the author gives an existence result for two-dimensional oscillatory Stokes flows past rigid obstacles. The corresponding stream function of the flow is represented in terms of simple layer potentials. The problem is reduced to an integral equations system of the first kind for which an existence result is established.*

Mathematical model of the flow

The equations corresponding to the motion of a viscous incompressible fluid past arbitrarily two-dimensional obstacles are the Narier - Stokes equations. By Ω_i we denote these obstacles with the boundaries C^i, $i=\overline{1,N}$, and in the following we suppose that $N \geq 2$. The fluid velocity $\vec{u}$ satisfies the boundary conditions:

(1) $$\vec{u}(x) = \vec{f}^i(x), \quad x \in C^i, \ i = \overline{1,N}$$

where

(2) $$\int_{C^i} \vec{f}^i(x) \cdot \vec{n}^i(x)\, ds^i = 0, \quad i = \overline{1,N},$$

and

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(3) $$\vec{f}^i \in C^{1+\gamma}(C^i)\ ,\quad i=\overline{1,N},\ 0<\gamma<1$$

We denote by $\vec{n}^i$ the unit outward normal vector to C^i and by $C^{1+\gamma}(C^i)$ the class of Hölder continously differentiable functions on C^i, $i=\overline{1,N}$, with the exponent $0 < \gamma < 1$.

The Reynolds number of the flow is supposed to be very small.

If we assume that the motion depends on the time t, then the nonstationary, unsteady Narier - Stokes equations are reduced to the dimensionless equations:

(4) $$\frac{\partial \vec{u}}{\partial t} = - \nabla p + \Delta \vec{u} \quad in\ \Omega$$

(5) $$\nabla \cdot \vec{u} = 0 \qquad in\ \Omega$$

where Ω denotes the fluid domain and p the fluid pressure.

We consider that the velocity $\vec{u}$ satisfies the boundary conditions (1) (for simplicity, in the dimensionless case, we suppose that the boundary conditions have the same form as in (1)) and also the asymptotic conditions:

(6) $$\vec{u} \to \vec{I}\ ,\ p \to 0\ ,\ as\ |\vec{x}| \to \infty\ ,$$

where the fluid flow at infinity is considered to be uniform flow, and $|\vec{x}|=|x|=\sqrt{x_1^2+x_2^2}$ denotes the norm of the vector $\vec{x}$.

The functions which appear in (4) - (6) are periodic with the same period T>0, and have the form:

(7) $$\vec{u}(x,t) = Re\,\{\vec{U}(x)\,e^{-iTt}\},\ p(x,t) = \\ = Re\,\{P(x)\,e^{-iTt}\},\ \vec{f}^i(x,t) = Re\,\{\vec{F}^i(x)\,e^{-iTt}\},$$

$i=\overline{1,N}$, where Re z means the real part of the complex number z.

From (1), (4)-(5) and (7) we obtain the following oscillatory Stokes problem:

(8.a) $$\Delta \vec{U} - \nabla P = -iT\vec{U} \quad in\ \Omega$$

(8.b) $$\nabla \cdot \vec{U} = 0 \qquad in\ \Omega$$

(8.c) $$\vec{U}(x) = \vec{F}^i(x), \quad x \in C^i, \ i=\overline{1,N}.$$

The condition (6) at infinity is replaced by the next condition:

(8.d) $$\vec{U}(x) = O(|\vec{x}|^{-2}), \quad as \ |\vec{x}| \to \infty.$$

From (2) and (7) it results that the functions $\vec{F}^i, i=\overline{1,N}$ satisfy the conditions:

(8.e) $$\int_{C^i} \vec{F}^i(x) \cdot \vec{n}^i(x)\, ds^i = 0, \ \vec{F}^i \in C^{1+\gamma}(C^i), \ i=\overline{1,N}.$$

The continuity equation (8.b) gives the stream function ψ_k, such that:

(9) $$\vec{U} = (\nabla\psi_k)^{\perp},$$

where $\vec{v}^{\perp}$ denotes the ortogonal vector of the vector $\vec{v}$.

In terms of stream function ψ_k the above problem is reduced to the problem:

(10.a) $$\Delta^2\psi_k + ik^2\Delta\psi_k = 0 \quad \text{in } \Omega$$

(10.b) $$\nabla\psi_k(x) = \vec{g}^i(x), \quad x \in C^i, \ i=\overline{1,N}$$

(10.c) $$\nabla\psi_k(x) = O(|\vec{x}|^{-2}), \quad as \ |\vec{x}| \to \infty$$

where $k^2=T$, $\vec{g}^i=(\vec{F}^i)^{\perp}$, $i=\overline{1,N}$.

Aditionally, we suppose the following conditions at infinity:

(10.d) $$\psi_k(x) = O(|\vec{x}|^{-1}), \ D^m\psi_k(x) = O(|\vec{x}|^{-2}), \ m\geq 1, \ as|\vec{x}| \to \infty.$$

By D^m we denote any derivative of the order $m \geq 1$.

Let G_k be the fundamental solution of the equation (10.a), given by [1]:

(11) $$G_k(x,y) = J(|\vec{x}-\vec{y}|)$$

where $J(u) = 4i\int_0^u s^{-1}\left(\int_0^s \tau k_0(\sqrt{-i}k\tau)\, d\tau\right) ds.$

Here K_0 is the Bassel function of the order zero, which has a logarithmic singularity at the point $u = 0$.

Explicitely G_k has the form:

(12) $G_k(x,y) = F(x,y) + |x - y|^2 \ln k + H(x,y)$,

where F is the fundamental solution of the biharmonic equation, given by:

(13) $$F(x,y) = \frac{1}{8\pi}|x-y|^2\left[\ln|x-y|-1\right].$$

The function H has continous third derivatives and $H(x,y) = O(k^2 \ln k)$ as $k \to 0$, uniformly for x,y in compact sets.

We seek a solution of the problem (10.a) - (10.d) in the form:

(14) $$\psi_k(x) = \sum_{j=1}^{N}\int_{C^j}\nabla_y G_k(x,y)\cdot\phi^j(y)\,ds_y^j\ ,\ x\in\Omega,$$

where the symbol ∇_y means the gradient operator in respect to the variable y.

If we use the asymptotic development of the Bessel function K_0:

(15) $$K_0(u) = u^{-1/2}\cdot e^{-u}f_0(u),$$

with $|f_0(u)| \le c$ (constant) for $|u| > R$, and $|\arg u| \le \frac{\pi}{2}$, where R is sufficiently large, then it results that [1]:

(16) $$\nabla_y G_k(x,y) = -C\frac{\vec{x}}{|\vec{x}|^2} + O(e^{-k|\vec{x}|/\sqrt{2}}),\ as\ |\vec{x}| \to \infty,$$

for $|y|$ bounded.

Because G_k is the fundamental solution of the equation (10.a), then the function ψ_k, given by (14), satisfies this equation, and from the estimate (16), it results that the asymptotic conditions (10.d) are satisfied. The function ψ_k will be a solution of (10.b) when the density function $\phi: \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2$, $\phi|_{C^j} = \vec{\phi}^j, j=\overline{1,N}$, is a solution for the following integral system of the first kind:

(17) $$\sum_{i=1}^{N}\int_{C^i}\nabla_x\nabla_y G_k(x^j,y)\,\vec{\phi}^i(y)\,ds_y^i = \vec{g}^j(x),\ x\in C^j,\ j=\overline{1,N}.$$

If we differentiate in (17) with respect to the arc length s_x^j, $j=\overline{1,N}$, then, we obtain the following singular integral system, which has a Cauchy-type singularity and the index zero:

(18) $$\sum_{i=1}^{N}\int_{C^i}\frac{\partial}{\partial s_x^j}\nabla_x\nabla_y G_k(x^j,y)\cdot\vec{\phi}^i(y)\,ds_y^i = \frac{d}{ds_x^j}\vec{g}^j(x)\,,x\in C^j,\ j=\overline{1,N}.$$

The adjoint homogeneous system corresponding to the system (18) has the form:

(19) $$\sum_{i=1}^{N}\int_{C^i}\frac{\partial}{\partial s_x^i}\nabla_x\nabla_y G_k(x^i,y)\,\vec{\Phi}^i(x)\,ds_x^i = 0,\ \ y\in C^j,\ j=\overline{1,N}.$$

We observe that each function $\phi:\bigcup_{j=1}^{N}C^j\to\mathbf{R}^2$, given by:

(20) $$\phi(x)=a_i\vec{x}+\vec{b}_i\ ,\ x\in C^i,\ i=\overline{1,N}$$

represents a solution for the system (19), where a_i is a real constant and $\vec{b}_i$ is a constant vector, $i=\overline{1,N}$. The functions which have the linear form (20) determine a 3N dimensionaly real space.

Then the dimension of the solution's space for the system (19), and hence for the homogeneous system (18), is at least 3N.

We used here the Fredholm first alternative [7] which implies that the dimension of solution's space for the homogeneous system (18) is equal with the dimension of solution's space, corresponding to the adjoint system (19).

The next result shows that this dimension is exactly 3N.

Theorem 1. The homogeneous system corresponding to the system (18) has at most 3N lineary independent solution.

Proof. It is easily to show that the functions $\vec{\tau}^i:\bigcup_{j=1}^{N}C^j\to\mathbf{R}^2$, $i=\overline{1,N}$, are solutions of the homogeneous system (18), where:

(21) $$\vec{\tau}^i(x)=\begin{cases}0, & x\in C^j,\ j\neq i\\ \vec{\tau}^i, & x\in C^i\end{cases}$$

Here $\vec{\tau}^i$ means the unit tangent vector of C^i, at the point $x\in C^i$, $i=\overline{1,N}$.

Let $\varphi^j:\bigcup_{i=1}^{N}C^i\to\mathbf{R}^2$, $j=\overline{1,2N+1}$ any 2N+1 solutions of the homogeneous system (18). The corresponding stream functions $\psi_k^j, j=\overline{1,2N+1}$, satisfy the Stokes problems:

(22.a) $$\Delta^2\psi_k^j(x) + ik^2\Delta\psi_k^j(x) = 0, \ x\in\Omega$$

(22.b) $$\nabla\psi_k^j(x) = \vec{c}_j^{\,l}, \quad x\in C^l, \ l=\overline{1,N}$$

(22.c) $$\nabla\psi_k^j(x) \to 0, \quad as \ |\vec{x}| \to \infty$$

where $\vec{c}_j^{\,l}$, $l=\overline{1,N}$, $j=\overline{1,2N+1}$ are constant vectors.

We can determine the constants $\beta_1,\ldots,\beta_{2N+1}$ not all equal to zero, such that:

(23) $$\sum_{j=1}^{2N+1}\beta_j\vec{c}_j^{\,l} = 0, \ l=\overline{1,N}.$$

Then the function $\psi_k^0=\sum_{j=1}^{2N+1}\beta_j\psi_k^j$ must satisfies the following Stokes problem:

(24.a) $$\Delta^2\psi_k^0(x)+ik^2\Delta\psi_k^0(x) = 0, \ x\in\Omega$$

(24.b) $$\nabla\psi_k^0(x) = 0 \quad , \ x\in C^l, \ l=\overline{1,N}$$

(24.c) $$\nabla\psi_k^0(x) \to 0 \quad , \ as \ |\vec{x}|\to\infty$$

This problem has a constant solution. From the next uniqueness result, it follows that it is the unique solution of the problem (24.a) - (24.c).

Theorem 2. The Stokes equation (10.a) with the boundary and asymptotic conditions (10.b) - (10.d), and with the additionaly conditions:

(25) $$\int_{C^j}\frac{\partial\omega_k}{\partial n}(x)\,ds^j = ik^2\int_{C^j}\vec{g}^{\,j}(x)\cdot\vec{n}^{\,j}(x)\,ds^j, \ j=\overline{1,N},$$

has at most one solution, where $\omega_k = \Delta\psi_k$.

Proof. Let $\Omega_R = \Omega \cap B_R$, where B_R is a large disk of radius R such that all the domains Ω^i, $i=\overline{1,N}$, are included in Ω_R. From Green's formula, we obtain:

$$0=\int_{\Omega_R}\bar{\psi}_k(\Delta^2\psi_k+ik^2\Delta\psi_k)\,dx=\int_{\Omega_R}\Big[|\Delta\psi_k|^2-ik^2|\nabla\psi_k|\Big]dx+ \tag{26}$$
$$+\int_{\partial\Omega_R}\Big[\bar{\psi}_k\frac{\partial\omega_k}{\partial n}-\omega_k\frac{\partial\bar{\psi}_k}{\partial n}+ik^2\frac{\partial\psi_k}{\partial n}\Big]ds,$$

where ψ_k is a solution of the Stokes equation (10.a). If ψ_k^1 and ψ_k^2, are two solutions of Stokes problem (10.a) - (10.d), together the conditions (25), then the function $\psi_k=\psi_k^1-\psi_k^2$, represents a solution of the Stokes equation (10.a), with the homogeneous boundary conditions, the conditions (10.d) at infinity, and the conditions:

$$\int_{C^i}\frac{\partial\omega_k}{\partial n}(x)\,ds^i=0\ ,\ i=\overline{1,N}.$$

From the above arguments we deduce that all integrals along's $\partial\Omega_R$ (the boundary of the domain Ω_R) are zero, when $R\to\infty$. Hence $\nabla\psi_k = 0$ in Ω and $\psi_k^1=\psi_k^2$ (up to an additive constant).

Using the Theorem 2, it results that the function ψ_k^0 of the constant form is the unique solution of (24.a) - (24.c).

On the other hand the function $\psi_k^0=\sum_{j=1}^{N}\beta_j\psi_k^j$ satisfies the problem (24.a) - (24.b) in each domain Ω^i, $i=\overline{1,N}$, and it is continous together with its first derivatives on C^i, $i=\overline{1,N}$. With the same arguments as in the Theorem 2, we obtain that ψ_k^0 has a constant form in Ω^i, $i=\overline{1,N}$.

Let $\Phi:\bigcup_{j=1}^{N}C^j\to\mathbf{R}^2$, be the function corresponding to the stream function ψ_k^0:

$$\Phi=\sum_{i=1}^{2N+1}\beta_i\varphi^i\ ,\ \Phi|_{C^j}=\Phi^j,\ j=\overline{1,N}. \tag{27}$$

From the properties of simple layer potentials [1], we deduce the following relations:

(28.a) $$\left(\Delta\psi_k\right)^+(x) - \left(\Delta\psi_k\right)^-(x) = a_j\vec{\varphi}^j(x)\cdot\vec{n}^j(x)\ ,\ x\in C^j,\ j=\overline{1,N}$$

(28.b) $$\left(\frac{\partial}{\partial n^j}\Delta\psi_k\right)^+(x) - \left(\frac{\partial}{\partial n^j}\Delta\psi_k\right)^-(x) = a_j\frac{d}{ds^j}\left(\vec{\tau}^j\cdot\vec{\varphi}^j\right)(x)\ ,\ x\in C^j, j=\overline{1,N}$$

where the symbols "+" and "-", means the limits at the point $x\in C^j$, from Ω and Ω^j respectively, and a_j is a real constant, $j=\overline{1,N}$.

Because the function ψ_k^0 above defined, has a constant form in Ω and in each domain Ω^i, $i=\overline{1,N}$, then from (28.a) - (28.b), it results that there exist the constants $\alpha_i\in\mathbf{R}$, $i=\overline{1,N}$ such that:

(29) $$\vec{\Phi}^i(x) = \alpha_i\vec{\tau}^i,\ for\ x\in C^i,\ i=\overline{1,N}.$$

Using (27) and (29) we obtain:

(30) $$\sum_{i=1}^{2N+1}\beta_i\varphi^i(x) - \sum_{i=1}^{N}\alpha_i\vec{\tau}^i(x) = 0,\ x\in\bigcup_{j=1}^{N}C^j,$$

where the function $\vec{\tau}^i, i=\overline{1,N}$ are defined in (21). It is clear that the functions φ^i, $i=\overline{1,2N+1}$ and $\vec{\tau}^j, j=\overline{1,N}$, are linearly dependent, so, the proof of Theorem 1 is finished.

From the Fredholm second alternative [7], we deduce that the system (18) has a solution if and only if each solution of the adjoint system (19) is orthogonal to the function $\vec{g}:\bigcup_{j=1}^{N}C^j\to\mathbf{R}^2$ given by:

(31) $$\vec{g}(x) = \frac{d}{ds_x^i}\vec{g}^i(x)\ \ ,\ for\ x\in C^i\ ,\ i=\overline{1,N}.$$

Since any solution of the system (19) has a linear form (20), the above condition is satisfied.

Let $\Phi^0:\bigcup_{j=1}^{N}C^j\to\mathbf{R}^2$, be the solution of the system (18), such that $\Phi^0|_{C^j} = \vec{\Phi}^{0j}$, $j=\overline{1,N}$. Then the function $\psi_k^0=\psi_k^0(\Phi^0)$ satisfies:

(32.a) $$\Delta^2\psi_k^0(x) + ik^2\Delta\psi_k^0(x) = 0\ \ ,\ x\in\Omega$$

(32.b) $$\nabla\psi_k^0(x) = \vec{g}^j(x) + \vec{K}^j\ ,\ x\in C^j,\ j=\overline{1,N}$$

(32.c) $$\nabla\psi_k^0(x) \to 0\ ,\quad as\ |\vec{x}| \to \infty$$

where $\vec{K}^j$, $j=\overline{1,N}$ are constant vectors.

We consider the function $K^0:\bigcup_{j=1}^{N} C^j \to \mathbb{R}^2$, be defined by:

(33) $$K^0(x) = \vec{K}^j\ ,\ for\ x\in C^j\ ,\ j=\overline{1,N}.$$

Let $\Phi^1,\ldots,\Phi^{2N}$ and Ψ^i, $i=\overline{1,N}$, 3N linearly independent solutions of the homogeneous system (18). The corresponding stream functions $\Psi_k^j=\Psi_k^j(\Phi^j)$, $j=\overline{1,2N}$, are solutions for the next problems:

(34.a) $$\Delta^2\Psi_k^j(x) + i\Delta\Psi_k^j(x) = 0\ ,\ x\in\Omega$$

(34.b) $$\nabla\Psi_k^j(x) = \vec{c}_j^{\,l}\ ,\ x\in C^l\ ,\ l=\overline{1,N}$$

(34.c) $$\nabla\Psi_k^j(x) \to 0\ ,\quad as\ |\vec{x}| \to \infty$$

where $\vec{c}_j^{\,l}$, $l=\overline{1,N}$, $j=\overline{1,2N}$, are constant vectors.

We define the functions $c^j:\bigcup_{l=1}^{N} C^l \to \mathbb{R}^2$, $j=\overline{1,2N}$, given by:

(35) $$c^j(x) = \vec{c}_j^{\,l}\ ,\ for\ x\in C^l,\ l=\overline{1,N}.$$

From Theorem 2 and the relations (28.a) - (28.b) it is easily to prove that the functions c^j, $j=\overline{1,2N}$, are linearly independent. Also, we observe that these functions and the function K^0, defined by (33), belong to the 2N dimensional space:

(36) $$S=\left\{K:\bigcup_{j=1}^{N} C^j\to\mathbb{R}^2 \,|\, K(x)=\vec{c}^{\,j},\ for\ x\in C^j,\ j=\overline{1,N}\right\},$$
$\vec{c}^{\,j}$ is a constant vector, $j=\overline{1,N}$.

Hence, there exist the numbers $\gamma_i\in\mathbb{R}$, $i=\overline{1,2N}$, such that:

(37) $$\sum_{i=1}^{2N}\gamma_i c^j(x) + K^0(x) = 0,\ x\in\bigcup_{j=1}^{N} C^j.$$

From (32.a) - (32.c), (33), (34.a) - (34.c), (35) and (37), we deduce that the function ψ defined by:

(38) $$\psi_k = \sum_{i=1}^{2N}\gamma_i\Psi_k^i + \psi_k^0$$

is the unique solution of the oscillatory Stokes problem (10.a) -

(10.d), together the conditions (25).

So, using the above arguments, we can formulate the following result.

Theorem 3. If the functions $\vec{F}^i$, $i=\overline{1,N}$ satisfy the conditions (8.e), then there exists an oscillatory stream function ψ_* (unique up to an additive constant) which is a solution of the problem (10.a) - (10.d), with the additionaly conditions (25).

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