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THE NUMERICAL TREATMENT OF DELAY DIFFERENTIAL EQUATIONS WITH CONSTANT DELAY BY NATURAL SPLINE FUNCTIONS OF EVEN DEGREE

G. MICULA, P. BLAGA AND H. AKÇA

ABSTRACT. The natural spline functions of even degree by using the derivative–interpolating conditions are introduced. Such kind of spline functions are very suitable for the numerical solutions of delay differential equation problems.

1. INTRODUCTION

At the end of the twentieth century, which sometimes is called the century of information processing, the numerical treatment of differential equation problems has become increasingly interested in the area of applied mathematics. Spline theory is nowadays a very active field of approximation theory and many advantages exist for all its basic problems, which have been reconsidered. These include, for example, data fitting, function approximation, numerical integration and differentiation, numerical solution of operator equations, optimal control problems, calculation of eigenvalues and eigenfunctions of operators, computing aided geometric design, numerical methods of probabilities and statistics, etc. For detailed discussions on spline functions we refer to the monographs [9, 16] and for an exhaustive literature on spline functions and their applications we refer to [12]. Almost all papers underline the fundamental properties of natural spline functions of odd degree, namely the minimum norm property, the best approximation property, the convergence properties, etc.

Very recently in [3], by changing the interpolation conditions, it was defined a natural spline function of even degree which keeps all the remarkable properties of odd degree one and it was developed a very efficient procedure to use this spline for the numerical solution of differential equations with initial conditions. For details of the theory of spline functions of even degree we refer to [2].

In this paper we shall extend the method of spline of even degree to the numerical treatment of constant delay differential equations with initial conditions. The mean idea is to reduce the delay differential equation problem step by step to the ordinary initial value problem one.

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2. Basic definitions and properties

Let $[a, b]$ be a finite closed interval of the real axis, and let

$$\Delta_n := \{x_i\}_1^n, \quad \text{with } a = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n < x_{n+1} = b$$

be a partition of it in $n+1$ subintervals

$$I_k := [x_k, x_{k+1}[, \quad k = 0, 1, \ldots, n.$$

Let m be a given positive integer.

Definition 1. ([2]). The function $s : [a, b] \to \mathbb{R}$ is called *the natural spline function of degree* $2m$ if:

1^o $s \in C^{2m-1}[a, b]$,
2^o $s|_{I_k} \in \mathcal{P}_{2m}, \quad k = \overline{1, n-1}, \ s|_{I_0} \in \mathcal{P}_m, \ s|_{I_n} \in \mathcal{P}_m,$
where $\mathcal{P}_k$ is the set of polynomials of degree $\le k$.

We call the space

$$\begin{aligned} \mathcal{S}_{2m}(\Delta_n) = \{\, s : \ & \text{there exists polynomials } s_0, s_1, \ldots s_n \\ & \text{such that } s(x) = s_i(x) \text{ for } x \in I_i, \ i = 0, 1, \ldots, n; \\ & D^j s_{i-1}(x_i) = D^j s_i(x_i), \text{ for } j = 0, 1, \ldots, 2m-1 \,\} \end{aligned}$$

the space of natural polynomial splines of even degree $2m$ *with the simple knots* $x_1, x_2, \ldots, x_n$.

The space $\mathcal{S}_{2m}(\Delta_n)$ of splines is a subset of $C^{2m-1}[a, b]$.

Theorem 1. ([2]). *$\mathcal{S}_{2m}(\Delta_n)$ is a linear space of dimension $n+1$. Any element $s \in \mathcal{S}_{2m}(\Delta_n)$ has the following representation*

$$s(x) = \sum_{i=0}^{m} A_i x^i + \sum_{k=1}^{n} a_k (x - x_k)_+^{2m}, \tag{1}$$

where the real coefficients $(A_i)_0^m$ are arbitrary, and the coefficients $(a_k)_1^n$ satisfy the conditions

$$\sum_{k=1}^{n} a_k x_k^i = 0, \quad \text{for} \quad i = \overline{0, m-1}, \tag{2}$$

Definition 2. ([2]). Let $n \ge m$ be two given positive integer numbers and $Y \in \mathbb{R}^{n+1}$, $Y := (y_\alpha, y_1', \ldots, y_n')$ a given vector. The spline function $s \in \mathcal{S}_{2m}(\Delta_n)$ is called *the derivative-interpolating spline function for the vector* Y if

$$\begin{aligned} s(x_\alpha) &= y_\alpha, \quad x_\alpha \text{ is a given point from } [a, b], \\ s'(x_i) &= y_i', \quad i = \overline{1, n}. \end{aligned} \tag{3}$$

Let denote the derivative–interpolating spline for a given Y by s_Y.

Theorem 2. *([2]). Let $n \geq m$ and the vector $Y = (y_\alpha, y'_1, \ldots, y'_n)$ be given. Then there exists, and it is unique, a derivative–interpolating function s_Y in $\mathcal{S}_{2m}(\Delta_n)$.*

Corollary 1. *If $f : [a,b] \to \mathbb{R}$ is a given function for that the values $f(x_\alpha)$ and $f'(x_k)$, $k = \overline{1,n}$, are known, there exists and it is unique, a natural spline function $s_f \in \mathcal{S}_{2m}(\Delta_n)$ which is derivative-interpolating for f, such that s_f satifies the conditions:*

$$\begin{aligned} s'_f(x_k) &= f'(x_k), \quad k = \overline{1,n}, \\ s_f(x_\alpha) &= f(x_\alpha), \quad \text{for } x_\alpha \text{ fixed from } [a,b]. \end{aligned}$$

Corollary 2. *There exists a unique set of $n+1$ fundamental natural polynomial spline functions $s_k \in \mathcal{S}_{2m}(\Delta_n)$, $k = \overline{1,n}$, and $s_\alpha \in \mathcal{S}_{2m}(\Delta_n)$ satisfying the following conditions:*

$$\begin{aligned} s'_k(x_i) &= \delta_{ik}, \quad i,k = \overline{1,n}, \quad s_k(x_\alpha) = 0, \quad k = \overline{1,n}, \\ s_\alpha(x_\alpha) &= 1, \quad s'_\alpha(x_k) = 0, \quad k = \overline{1,n}, \quad x_\alpha \text{ fixed from } [a,b]. \end{aligned}$$

It is clear that the functions s_α, s_k, $k = \overline{1,n}$, form a basis of the linear space $\mathcal{S}_{2m}(\Delta_n)$, and for s_f we have the following representation

$$s_f(x) = s_\alpha(x) f(x_\alpha) + \sum_{k=1}^{n} s_k(x) f'(x_k). \tag{4}$$

If $m = n$ it follows that $s_f \in \mathcal{P}_m$.

For the sake of its necessity let us introduce the following sets of functions

$$W_2^{m+1}(\Delta_n) := \left\{ f : [a,b] \to \mathbb{R} \,|\, f^{(m)} \text{ is abs. cont. on each } I_k \text{ and } f^{(m+1)} \in L_2[a,b] \right\},$$

$$W_{\substack{0\\2,Y}}^{m+1}(\Delta_n) := \left\{ f \in W_2^{m+1}(\Delta_n) \,|\, f'(x_k) = y'_k, \quad k = \overline{1,n} \right\},$$

$$W_2^{m+1}[a,b] := \left\{ f : [a,b] \to \mathbb{R} \,|\, f^{(m)} \text{ is abs. cont. on [a,b] and } f^{(m+1)} \in L_2[a,b] \right\},$$

and let denote

$$J(f) := \int_a^b \left(f^{(m+1)}(x) \right)^2 dx, \quad f \in W_{\substack{0\\2,Y}}^{m+1}(\Delta_n).$$

Theorem 3. **(Minimal norm property).** *([2]). If $s \in \mathcal{S}_{2m}(\Delta_n) \cap W_{\substack{0\\2,Y}}^{m+1}(\Delta_n)$, then*

$$J(s) = \min \left\{ J(f) \,:\, f \in W_{\substack{0\\2,Y}}^{m+1}(\Delta_n) \right\}.$$

With the usual notations this theorem asserts that

$$\|s^{(m+1)}\|_2^2 \leq \|f^{(m+1)}\|_2^2, \quad f \in W_{\substack{0\\2,Y}}^{m+1}(\Delta_n), \quad \|\cdot\|_2 \text{ being the } L_2\text{-norm.}$$

Theorem 4. (Best approximation property). *([2]). Let $f \in W_2^{m+1}(\Delta_n)$ be a given function and $s_f \in S_{2m}(\Delta_n)$ the natural derivative–interpolating spline function of even degree for f. For any $s \in S_{2m}(\Delta_n)$ holds:*

$$\|s_f^{(m+1)} - g^{(m+1)}\|_2^2 \le \|s^{(m+1)} - g^{(m+1)}\|_2^2,$$

where $g \in \overset{\circ}{W}{}^{m+1}_{2,f}(\Delta_n) := \{ h \in W_2^{m+1}(\Delta_n) \mid h'(x_k) = f'(x_k), \quad k = \overline{1,n} \}$.

Remark. In the above relation the equality holds if and only if $s_f - s \in \mathcal{P}_m$.

3. The numerical solutions of delay differential equations by spline functions of even degree

One considers the following delay differential equation problem

$$\begin{cases} y'(x) = f(x, y(x), y(x-\omega)), & x \in [a,b], \quad \omega > 0, \\ y(x) = \varphi(x), & x \in [a-\omega, a] \end{cases} \tag{5}$$

where $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ satisfies all the conditions assuring the existence and uniqueness of the solution $y : [a,b] \to \mathbb{R}$ of the problem (5).

We develop an algorithm to approximate the solution y of (5) by a spline function $s \in S_{2m}(\Delta_n)$, where Δ_n is a partition on each subinterval I_k of the length ω, and m, n are two integers satisfying the conditions $n \ge 2$ and $n \ge m > 0$.

On the first interval $a \le x \le a + \omega$, the problem (5) becomes

$$\begin{cases} y'(x) = f(x, y(x), \varphi(x-\omega)) =: g(x, y(x)), & a \le x \le a+\omega, \\ y(a) = \varphi(a), \end{cases} \tag{6}$$

which is an usual initial value problem.

Taking a partition $\Delta_n : a = x_0 < x_1 < \cdots < x_n < x_{n+1} = a + \omega$ of the interval $[a, a+\omega]$, and using all the arguments of [3], for the initial value problem, we have the following theorem:

Theorem 5. *If y is the exact solution of the problem (6) then there exists, and it is unique, a spline function of even degree $s_y \in S_{2m}(\Delta_n)$ such that:*

$$\begin{aligned} s_y'(x_k) &= y'(x_k), \quad k = \overline{1,n}, \\ s_y(x_0) &= y(x_0) = y(a) = \varphi(a), \end{aligned} \tag{7}$$

hold.

The assertion of this theorem is a direct consequence of Corollary 1 of the Theorem 2 by substituting x_α with x_0 and f with y.

If we denote $y_k := y(x_k)$, $k = \overline{1,n}$, then (7) implies:

$$\begin{aligned} s_y'(x_k) &= g(x_k, y_k), \quad k = \overline{1,n}, \\ s_y(x_0) &= y_0 = \varphi(a). \end{aligned}$$

Corollary 1. *If the functions $s_0, s_1, \dots, s_n \in \mathcal{S}_{2m}(\Delta_n)$ are satisfying the conditions*

$$s_0(x_0) = 1, \quad s_0'(x_k) = 0, \quad k = \overline{1,n},$$
$$s_i(x_0) = 0, \quad s_i'(x_k) = \delta_{ik}, \quad k = \overline{1,n}, \quad i = \overline{1,n},$$

then the spline function s_y has the representation

$$s_y(x) = y_0 + \sum_{k=1}^{n} s_k(x)\, g(x_k, y_k), \quad x \in [a, a+\omega], \tag{8}$$

where the unknowns y_k, $k = \overline{1,n}$, are to be determined as in [3].

Moreover, the following estimation error and convergence theorem is true:

Theorem 6. *([3]). If y is the exact solution of the problem (6), $y \in W_2^{m+1}[a,b]$, and s_y is the spline approximating solution (8), then the following estimations*

$$\|y^{(k)} - s_y^{(k)}\|_\infty \le \sqrt{m}\,(m-1)(m-2)\dots k\|\Delta_n\|^{m-k+\frac{1}{2}}\|y^{(m+1)}\|_2 \tag{9}$$

hold, for $k = 1, 2, \dots, m$, and $\|\Delta_n\| := \max\limits_{i=\overline{0,n}}\{\, x_{i+1} - x_i \,\}$.

Corollary 1. *If the exact solution of the problem (6) $y \in W_2^{m+1}[a,b]$, and s_y is the spline approximating solution then*

$$\|y - s_y\|_\infty \le (b-a)\sqrt{m}\,(m-1)!\;\|\Delta_n\|^{m-\frac{1}{2}}\left\|y^{(m+1)}\right\|_2$$

holds.

Corollary 2. *Under the same assumptions hold*

$$\lim_{\|\Delta_n\|\to 0}\left\|y^{(k)} - s_y^{(k)}\right\|_\infty = 0, \quad \textit{for} \quad k = 0, 1, \dots, m,$$

and the order of convergence is $m - k + \frac{1}{2}$.

As it was shown in [3], to construct effectively the approximating spline function (8), the values $s_y(x_i)$ are used, considering that $s_y(x) \approx y(x)$, $x \in [a, a+\omega]$. Writing for simplicity the spline approximating solution by $s(x)$, and denoting $s(x_i) =: w_i$, $i = \overline{1,n}$, we have to find the solution $w_1, w_2, \dots, w_n$ of the nonlinear system

$$w_i = y_0 + \sum_{k=1}^{n} s_k(x_i)\, g(x_k, w_k), \quad i = \overline{1,n}, \tag{10}$$

which converges to the exact solution y_i for $\|\Delta_n\| \to 0$.

Let us introduce the following matrix

$$A := \begin{pmatrix} s_1(x_1)\frac{\partial g(x_1,w_1)}{\partial w_1} & \dots & s_n(x_1)\frac{\partial g(x_n,w_n)}{\partial w_n} \\ \dots & \dots & \dots \\ s_1(x_n)\frac{\partial g(x_1,w_1)}{\partial w_1} & \dots & s_n(x_n)\frac{\partial g(x_n,w_n)}{\partial w_n} \end{pmatrix},$$

then the matrix A can be written as $A = S\,G$, where

$$S := \begin{pmatrix} s_1(x_1) & \dots & s_n(x_1) \\ \dots & \dots & \dots \\ s_1(x_n) & \dots & s_n(x_n) \end{pmatrix}, \quad G := \begin{pmatrix} \frac{\partial g(x_1,w_1)}{\partial w_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{\partial g(x_n,w_n)}{\partial w_n} \end{pmatrix}.$$

Now, for the solvability of the system (10), we have the following theorem.

Theorem 7. *Suppose that*

$$\left|\frac{\partial g(x,y)}{\partial y}\right| \le M, \quad \text{and} \quad |g(x,y)| \le N, \quad \forall (x,y) \in D.$$

If $M < \|S\|^{-1}$, then the system (10) has a solution which can be determined by iteration.

Proof. Applying the fixed-point theorem of Banach in the domain

$$\Omega = \left\{ (w_1, \dots, w_n) \in \mathbb{R}^n \;\middle|\; |w_i| \le |y_0| + N \sum_{k=1}^{n} |s_k(x_i)|, i = \overline{1,n} \right\},$$

for the system (10), because $M\,\|S\| < 1$, the assertion of theorem follows directly.

Now, going on the next interval $[a+\omega, a+2\omega]$, we shall use the same algorithm by using the spline function s_y obtained on the previous interval $[a, a+\omega]$, in the role of the function φ.

Generally, if $x \in [a+k\omega, a+(k+1)\omega]$, the problem (5) can be written as

$$\begin{cases} y'(x) = f(x, y(x), y(x-\omega)), & x \in [a+k\omega, a+(k+1)\omega], \\ y(x) = s_y^{[k]}(x), & x \in [a+(k-1)\omega, a+k\omega], \end{cases} \tag{11}$$

where $s_y^{[k]}$ is the spline function of even degree constructed by the above method on the previous interval $[a+(k-1)\omega, a+k\omega]$.

Taking again the partition Δ on the interval $[a+k\omega, a+(k+1)\omega]$

$$\Delta : a + k\omega = x_0 < x_1 + k\omega < \dots < x_n + k\omega < x_{n+1} + k\omega = a + (k+1)\omega,$$

the spline approximating function $s_y^{[k+1]}$ can be written effectively by:

$$s_y^{[k+1]}(x) = s_y^{[k]}(a+k\omega) + \sum_{j=1}^{n} s_j(x-k\omega)\, f\left(x_j + k\omega, y(x_j+k\omega), s_y^{[k]}(x_j+(k-1)\omega)\right).$$

Denoting $s_y^{[k+1]}(x_i + k\omega) =: w_i^{[k+1]} \approx y(x_i + k\omega)$, $i = \overline{1, n+1}$, the algebraic system (10) becomes

$$w_i^{[k+1]} = w_{n+1}^{[k]} + \sum_{j=1}^{n} s_j(x_i) f\left(x_j + k\omega, w_j^{[k+1]}, w_j^{[k]}\right), \quad i = \overline{1,n}. \qquad (12)$$

The system (12) is solving by iteration method, so that we can find the values $w_i^{[k+1]}$, $i = \overline{1,n}$, taking as starting values $w_{i0}^{[k+1]} = w_{n+1}^{[k]}$, $i = \overline{1,n}$.

Thus, the spline approximating function $s_y^{[k+1]}$ of even degree for the solution of the problem (5) on the interval $[a + k\omega, a + (k+1)\omega]$, has the following representation

$$s_y^{[k+1]}(x) = w_{n+1}^{[k]} + \sum_{j=1}^{n} s_j(x - k\omega) f\left(x_j + k\omega, w_j^{[k+1]}, w_j^{[k]}\right),$$

where s_j, $j = \overline{1,n}$ are the spline basis functions constructed on the first interval $[a, a+\omega]$.

4. Numerical examples

Consider the following delay differential equations with initial conditions, and the corresponding exact solutions. In the below tables the actual errors on each subinterval $I_k = [a + k\omega, a + (k+1)\omega]$, for the considered examples are given. Namely, these tables list

$$\max\left\{ |w_i^{[k+1]} - y(a + k\omega + x_i)|, \ i = \overline{1, n+1}; \right.$$
$$\left. |s_y^{[k+1]}(a + k\omega + 0.1\,i) - y(a + k\omega + 0.1\,i)|, \ i = \overline{0, 10\omega} \right\},$$

on the intervals I_k for the each three examples.

Example 1.

$$\begin{cases} y'(x) = y(x)\,y(x-1) - e^{2x-1} + e^x, & x \in [0,2], \\ y(x) = \varphi(x) = e^x, & x \in [-1,0]. \end{cases}$$
$$y(x) = e^x.$$

I_k	[0,1]			[1,2]		
n\m	1	2	3	1	2	3
9	.107−01	.130−02	.471−04	.138+00	.187−01	.645−03
19	.275−02	.163−03	.290−05	.361−01	.232−02	.371−04
29	.124−02	.485−04	.571−06	.166−01	.686−03	.738−05
39	.699−03	.205−04	.174−06	.951−02	.289−03	.228−05
49	.449−03	.105−04	.769−07	.616−02	.148−03	.101−05

Example 2.

$$\begin{cases} y'(x) = \dfrac{24x^2 y\,(x-\frac{1}{2})}{8y\,(x) - 12x^2 + 6x - 1}, & x \in [0,2], \\ y\,(x) = \varphi(x) = 1 + x^3, & x \in [-\frac{1}{2}, 0]. \end{cases}$$
$$y\,(x) = 1 + x^3.$$

I_k	[0,0.5]		[0.5,1]		[1,1.5]		[1.5,2]	
$n\backslash m$	1	2	1	2	1	2	1	2
9	.283−02	.406−03	.612−02	.623−03	.950−02	.807−03	.128−01	.966−03
19	.751−03	.507−04	.159−02	.789−04	.246−02	.102−03	.331−02	.122−03
29	.340−03	.150−04	.718−03	.235−04	.110−02	.305−04	.149−02	.364−04
39	.193−03	.633−05	.407−03	.992−05	.625−03	.129−04	.841−03	.154−04
49	.124−03	.324−05	.261−03	.508−05	.401−03	.662−05	.540−03	.790−05

Example 3.

$$\begin{cases} y'(x) = -y(x - \frac{\pi}{2}), & x \in [0, \frac{3\pi}{2}], \\ y\,(x) = \varphi(x) = \sin\, x, & x \in [-\frac{\pi}{2}, 0]. \end{cases}$$
$$y\,(x) = \sin\, x.$$

I_k	$[0,\frac{\pi}{2}]$			$[\frac{\pi}{2},\pi]$			$[\pi,\frac{3\pi}{2}]$		
$n\backslash m$	1	2	3	1	2	3	1	2	3
9	.926−02	.125−02	.109−03	.926−02	.176−02	.109−03	.826−02	.338−02	.927−04
19	.245−02	.150−03	.773−05	.245−02	.220−03	.773−05	.170−02	.426−03	.595−05
29	.111−02	.446−04	.158−05	.111−02	.650−04	.158−05	.711−03	.126−03	.135−05
39	.627−03	.188−04	.506−06	.627−03	.274−04	.506−06	.415−03	.533−04	.447−06
49	.403−03	.964−05	.212−06	.403−03	.140−04	.212−06	.271−03	.273−04	.195−06

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