LIBERTAS MATHEMATICA, VOL. XVI (1996)

EXISTENCE OF SOLUTIONS FOR AN INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION IN BANACH SPACES

J. P. Dauer and K. Balachandran

Abstract: The aim of this paper is to prove the existence and uniqueness of local and global solutions of a nonlocal Cauchy problem for an integrodifferential equation. The method of semigroups and the contraction mapping principle are used to establish the results.

1. Introduction

The problem of existence of solutions of evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [5]. In that work he has established the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + a], \tag{1}$$

$$u(t_0) + g(t_1, t_2, ..., t_p, u(.)) = u_0 \tag{2}$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t), t \geq 0$, on a Banach space $X, 0 \leq t_0 \leq t_1 < t_2 < ... < t_p \leq t_0 + a, a > 0, u_0 \in X$ and $f : [t_0, t_0 + a] \times$

$X \to X$, $g(t_1, t_2, ..., t_p, .) : X \to X$ are the given functions. Subsequently Byszewski has investigated the same type of problem for a different class of evolution equations in Banach space [2–4, 6-8]. Moreover, Cordaneanu [9] and Gripenberg et al. [10] studied the problem for Volterra integral equations of various types using a semigroup approach. The purpose of this paper is to prove the existence and uniqueness of a local solution for an integrodifferential equation with nonlocal conditions of the form

$$\frac{du(t)}{dt} + Au(t) = f\Big(t, u(t), \int_0^t k\big(t, s, u(s)\big)ds\Big) \tag{3}$$

$$u(0) + g\big(t_1, t_2, ..., t_p, u(\cdot)\big) = u_0 \tag{4}$$

Here it is assumed that $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operators $X(t),\ t \geq 0$, in a Banach space Z. The operator A^α can be defined for $0 \leq \alpha < 1$ and A^α is a closed linear invertible operator with domain $D(A^\alpha)$ endowed with the graph norm of A^α, that is the norm $\| z \|_0 = \| z \| + \| A^\alpha z \|$, is a Banach space. Since A^α is invertible its graph norm $\| \cdot \|_0$ is equivalent to the norm $\| z \|_\alpha = \| A^\alpha z \|$. Thus, $D(A^\alpha)$ equipped with the norm $\| \cdot \|_\alpha$ is a Banach space which is denoted by Z_α. From this definition it is clear that $0 < \alpha < \beta$ implies $Z_\alpha \supset Z_\beta$ and that the imbedding of Z_β in Z_α is continuous. Take $J = [0, a]$ and $\Delta = \{(t, s) : 0 \leq s < t \leq a\}$. The nonlinear operators $f : J \times Z_\alpha \times Z_\alpha \to Z, k : \Delta \times Z \to Z_\alpha$ and $g(t_1, t_2, ..., t_p, .) : Z_\alpha \to Z$ are given functions. The symbol $g\big(t_1, t_2, ..., t_p, u(\cdot)\big)$is used in the sense that in place of "." we can substitute only elements of the set $\{t_1, t_2, ..., t_p\}$.

The results obtained in this paper are generalizations of the known results given by Pazy [12] about the Cauchy problem. As in [1–3, 6,7,11] the nonlocal condition (2) in this paper can be applied in physics with better effect than the classical condition $u(0) = u_0$. This is because condition (2) is usually more precise for physical measurements than the classical condition.

2. Existence Theorems

Theorem 2.1. *Assume that the follwoing six conditions hold:*

(i) $-A$ *is the infinitesimal generator of a bounded analytic semigroup of linear operator* $X(t), t > 0$, in Z.

(ii) For $0 \leq \alpha < 1$, *the fractional power* A^α satisfies $\| A^\alpha X(t) \| \leq C_\alpha t^{-\alpha}$ *for* $t > 0$ *where* C_α *is a real constant.*

(iii) $0 \in \rho(-A)$, *the resolvent set.*

(iv) Let E *be an open set of* $J \times Z_\alpha \times Z_\alpha$ *and* $f : E \to Z$ *be such that for every* $(t, u, v) \in E$ *there is a neighborhood* $U \subset E$ *and constants* $L_1 \geq 0, 0 < \theta \leq 1$ *such that for all* $(t, u_i, v_i) \in U$, $i = 1, 2$

$$\| f(t, u_1, v_1) - f(s, u_2, v_2) \| \leq L_1 \big(|t - s|^\theta + \| u_1 - u_2 \|_\alpha + \| v_1 - v_2 \|_\alpha \big). \tag{5}$$

(v) Let V *is an open subset of* $\Delta \times Z$ *and* $k : V \to Z_\alpha$ *be such that for every* $(t, s, u) \in V$ *there is a neighborhood* $W \subset V$ *and constants* $L_2 \geq 0, 0 < \theta \leq 1$ *such that*

$$\| k(t, \tau, u_1) - k(s, \tau, u_2) \| \leq L_2 \big(|t - s|^\theta + \| u_1 - u_2 \|_\alpha \big). \tag{6}$$

(vi) $g : J^p \times Z_\alpha \to Z$ *and there exist constants* $B^* > 0$ *and* $L > 0$ *such that*

$$\| A^\alpha g(t_1, t_2, ..., t_p, u(\cdot)) \| \leq B^* \quad for\ 0 \leq t < a,$$

$$\| g(t_1, t_2, ...t_p, u_1(\cdot)) - g(t_1, t_2, ..., t_p, u_2(\cdot)) \| \leq L \| u_1 - u_2 \|_\alpha .$$

Then there is a unique local solution $u \in C([0,a) : Z) \cap C^1((0,a) : Z)$ *of the nonlocal Cauchy problem (3),(4).*

Proof. Choose $t^* > 0$ and $\delta > 0$ such that the estimates (5) and (6) hold in the sets $U = \{(t,u,v) : 0 \leq t \leq t^*, \| u - u_0 \|_\alpha \leq \delta, \| v - v_0 \|_\alpha \leq \delta\}$ and $V = \{(t,s,u) : 0 \leq s < t \leq t^*, \| u - u_0 \|_\alpha \leq \delta\}$, respectively. Let

$$H = \max_{0 \leq t \leq t^*} \| f\Big(t, u_0, \int_0^t k(t,s,u_0)ds\Big) \|,$$

$$L = \max_{0 \leq t,s \leq t^*} \| k(t,s,u_0) \|$$

and choose a constant a such that for $0 \leq t < a$

$$\| X(t)A^\alpha u_0 - A^\alpha u_0 \| < \delta/4,$$

$$\| X(t)A^\alpha g(t_1, t_2, ..., t_p, u(\cdot)) - A^\alpha g(t_1, t_2, ..., t_p, u(\cdot)) \| < \delta/4$$

$$0 < a < \min\Big\{t^*, \Big[(\delta/2)(1-\alpha)C_\alpha^{-1}(L_1\delta a + L_1 B^* a + L_1 L_2 \delta a^2 + L_1 L_2 B^* a^2 + Ha)^{-1}\Big]^{1/(1-\alpha)}\Big\}. \qquad (7)$$

Let Y be the Banach space $C((0,a] : Z)$ with usual supremum norm which is denoted by

$\|\cdot\|$. Define a map $F : Y \to Y$ by

$$Fy(t) = X(t)A^{\alpha}u_0 - X(t)A^{\alpha}g\big(t_1, t_2, ..., t_p, A^{-\alpha}y(\cdot)\big) + \\ + \int_0^t A^{\alpha}X(t-s)f\Big(s, A^{-\alpha}y(s), \int_0^s k\big(s,\tau,A^{-\alpha}y(\tau)\big)d\tau\Big)ds. \tag{8}$$

Obviously $Fy(0) = A^{\alpha}u_0 - A^{\alpha}g$. Let S be the nonempty closed and bounded subset of Y defined by

$$S = \{y : y \in Y, y(0) = A^{\alpha}u_0 - A^{\alpha}g, \| y(t) - (A^{\alpha}u_0 - A^{\alpha}g) \| \leq \delta\}.$$

For $y \in S$, it follows that

$$\begin{aligned}
\| Fy(t) - (A^{\alpha}u_0 - A^{\alpha}g) \| &\leq \| X(t)A^{\alpha}u_0 - A^{\alpha}u_0 \| + \\
&\quad \| X(t)A^{\alpha}g\big(t_1,t_2,...,t_p,A^{-\alpha}y(\cdot)\big) - A^{\alpha}g\big(t_1,t_2,...,t_p,A^{-\alpha}y(\cdot)\big) \| \\
&\quad + \| \int_0^t A^{\alpha}X(t-s)\Big[f\Big(s, A^{-\alpha}y(s), \int_0^s k\big(s,\tau,A^{-\alpha}y(\tau)\big)d\tau\Big) \\
&\quad - f\Big(s, u_0, \int_0^s k(s,\tau,u_0)d\tau\Big)ds\Big] \| \\
&\quad + \| \int_0^t A^{\alpha}X(t-s)f\Big(s, u_0, \int_0^s k(s,\tau,u_0)d\tau\Big)ds \| \\
&\leq \delta/4 + \delta/4 + \int_0^t \| A^{\alpha}X(t-s) \| \{L_1 \| A^{-\alpha}y(s) - u_0 \| \\
&\quad + L_2 \| A^{-\alpha}y(\tau) - u_0 \| a^2\}ds + C_{\alpha}a^*Ha \\
&\leq \delta/4 + \delta/4 + C_{\alpha}a^*\{L_1(\delta + B^* + L_2(\delta + B^*)a)\}a + C_{\alpha}a^*Ha \\
&\leq \delta/2 + C_{\alpha}a^*\{L_1\delta a + L_1B^*a + L_1L_2\delta a^2 + L_1L_2B^*a^2 + Ha\} \\
&\leq \delta/2 + \delta/2 = \delta, \ \textit{where} \ a^* = a^{(1-\alpha)}(1-\alpha)^{-1}.
\end{aligned}$$

Therefore, F map S into itself. More over, if $y_1, y_2 \in S$, then

$$\begin{aligned}
&\| Fy_1(t) - Fy_2(t) \| \\
&\quad \leq \| X(t)\Big(A^\alpha g(t_1, t_2, \dots t_p, A^{-\alpha} y_1(\cdot)) - A^\alpha g(t_1, t_2, \dots t_p, A^{-\alpha} y_2(\cdot))\Big) \| \\
&\qquad + \| \int_0^t A^\alpha X(t-s)\Big[f\Big(s, A^{-\alpha} y_1(s), \int_0^s k(s, \tau, A^{-\alpha} y_1(\tau)\Big) d\tau \\
&\qquad - f\Big(s, A^{-\alpha} y_2(s), \int_0^s k(s, \tau, A^{-\alpha} y_2(\tau)) d\tau\Big) ds\Big] ds \\
&\quad \leq C_\alpha a^* L \| y_1 - y_2 \| + C_\alpha a^* L_1 [\, \| y_1 - y_2 \| + L_2 \| y_1 - y_2 \| \, a)a] \\
&\quad \leq C_\alpha a^* [L + L_1(1 + L_2 a)a)] \| y_1 - y_2 \| \\
&\quad \leq \frac{1}{2} \| y_1 - y_2 \|
\end{aligned}$$

which implies that $\| Fy_1 - Fy_2 \| \leq \frac{1}{2} \| y_1 - y_2 \|$. By the contraction mapping theorem the mapping F has a unique fixed point $y \in S$. This fixed point satisfies the integral equation

$$\begin{aligned}
y(t) &= X(t)A^\alpha u_0 - X(t)A^\alpha g(t_1, t_2, \dots, t_p, A^{-\alpha} y(\cdot)) \\
&\quad + \int_0^t A^\alpha X(t-s) f\Big(s, A^{-\alpha} y(s), \int_0^s k(s, \tau, A^{-\alpha} y(\tau)) d\tau\Big) ds. \qquad (9)
\end{aligned}$$

From (5) and the continuity of y it follows that

$$t \to f\Big(t, A^{-\alpha} y(t), \int_0^t k(t, s, A^{-\alpha} y(s)) ds\Big)$$

is continuous on $[0, a]$, and hence there exists constant H^* such that

$$\| f\Big(t, A^{-\alpha} y(t), \int_0^t k\big(t, s, A^{-\alpha} y(s)\big) ds \| \leq H^*. \qquad (11)$$

Note that for every β satsifying $0 < \beta < 1 - \alpha$ and every $0 < h < 1$ it follows that [12, Theorem 2.6.13] there is a $C > 0$ such that

$$\| (X(h) - I)A^{\alpha}X(t-s) \| \leq C_{\alpha}h^{\beta} \| A^{\alpha+\beta}X(t-s) \|$$
$$\leq Ch^{\beta}(t-s)^{-(\alpha+\beta)}. \tag{12}$$

If $0 < t < t+h < a$, then

$$\begin{aligned}\| y(t+h) - y(t) \| \leq & \| (X(h) - I)A^{\alpha}X(t)u_0 \| \\ & + \| (X(h) - I)A^{\alpha}X(t)g\big(t_1, t_2, ..., t_p, A^{-\alpha}y(.)\big) \| \\ & + \int_0^t \| X(h) - I)A^{\alpha}X(t-s)f\Big(s, A^{-\alpha}y(s), \int_0^s k\big(s,\tau, A^{-\alpha}y(\tau)\big)d\tau\Big) \| ds \\ & + \int_t^{t+h} \| A^{\alpha}X(t+h-s)f\Big(s, A^{-\alpha}y(s),, \int_0^s k\big(s,\tau, A^{-\alpha}y(\tau)\big)d\tau\Big) \| ds \\ = & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{13}$$

Using (vi), (11) and (12) it follows that

$$I_1 \leq Ch^{\beta}t^{-(\alpha+\beta)} \| u_0 \| \leq M_1h^{\beta},$$

$$I_2 \leq Ch^{\beta}t^{-(\alpha+\beta)}B^* \leq M_2h^{\beta},$$

$$I_3 \leq Ch^{\beta}H^* \int_0^t (t-s)^{-(\alpha+\beta)}ds \leq M_3h^{\beta},$$

$$I_4 \leq H^*C_{\alpha}\int_t^{t+h}(t+h-s)^{-\alpha}ds = H^*C_{\alpha}(1-\alpha)^{-1}h^{(1-\alpha)} \leq M_4h^{\beta}.$$

Here M_1 and M_2 depend on t and vanish at $t \to 0$, but M_3 and M_4 can be selected to be independent of $t \in J$. Combining (13) with these estimates it follows that for every $t' > 0$ there is a constant C such that $\| y(t) - y(s) \| \le C|t-s|^{\beta}$ for $0 < t' < t, s \le a$. Therefore y is locally Hölder continuous on $(0,a]$. The local Hölder continuity of $t \to$

$f(t, A^{-\alpha}y(t), \int_o^t k(t,s,A^{-\alpha}y(s)ds)$ follows from

$$\| f\Big(t, A^{-\alpha}y(t), \int_0^t k(t,s,A^{-\alpha}y(\tau))d\tau\Big) - f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big) \|$$

$$\le L_1\{|t-s|^{\theta} + \| y(t) - y(s) \| + L_2(|t-s|^{\theta} + |t-s|(\delta + B^*))a\}$$

$$\le L_1(|t-s|^{\beta} + L_3(|t-s|^{\theta}) \textit{ for some } L_3 > 0.$$

Let y be the solution of (9) and consider the inhomogeneous initial value problem

$$\frac{du(t)}{dt} + Au(t) = f\Big(t, A^{-\alpha}y(t), \int_0^t k(t,s,A^{-\alpha}y(s))ds\Big) \tag{14}$$

$$u(0) + g\big(t_1, t_2, ..., t_p, A^{-\alpha}y(\cdot)\big) = u_0. \tag{15}$$

Then the problem has a unique solution $u \in C^1\big((0,a] : Z\big)$ which is given by

$$u(t) = X(t)u_0 - X(t)g\big(t_1, t_2, ..., t_p, A^{-\alpha}y(\cdot)\big) + \\ + \int_0^t X(t-s)f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)ds. \tag{16}$$

For $t > 0$ each term of (16) is in $D(A)$ and *a fortiori* in $D(A^{\alpha})$. Operating on both sides of (16) with A^{α} it follows that

$$A^{\alpha}u(t) = X(t)A^{\alpha}u_0 - X(t)A^{\alpha}g\big(t_1, t_2, ..., t_p, A^{-\alpha}y(.)\big) \\ + \int_0^t A^{\alpha}X(t-s)f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)ds. \tag{17}$$

From (9) and (17), $u(t) = A^{-\alpha}y(t)$ and by (16), u is a $C^1\big((0,a] : Z\big)$ solution of (3). The uniqueness of the solutions of (9) and (14), (15) gives the uniqueness of u. Hence the theorem is proved. □

Next the existence of global solutions of (3), (4) shall be proven.

Theorem 2.2. *Let $0 \in \rho(-A)$ and let $-A$ be the infintesimal generator of an analytic semigroup $X(t)$ satisfying $\| X(t) \| \leq M$ for $t \geq 0$. Let $f : I \times Z_\alpha \times Z_\alpha \to Z$ and $g(t_1, t_2, ..., t_p, u(.)) : I^p \times Z_\alpha \to Z_\alpha$ satisfy (5) and (6) respectively, with $I = [0, \infty)$. If there is a continuous nondecreasing real valued function $q(t)$ such that*

$$\| f\Big(t, u(t), \int_0^t k(t, s, u(s))ds\Big) \| \leq q(t)(1 + \| u \|_\alpha) \quad \text{for } t \geq 0, u \in Z_\alpha,$$

then for every $u_0 \in Z_\alpha$ the initial value problem (3), (4) has a unique solution u.

Proof. As in the proof of Theorem (2.1), the solution of (3) can be continued as long as $\| u(t) \|_\alpha$ remains bounded. It is enough to prove that if u exists on $[0, a)$ then $\| u(t) \|_\alpha$ is bounded as $t \to a$. Since

$$\begin{aligned} A^\alpha u(t) = {} & X(t)A^\alpha u_0 - X(t)A^\alpha g(t_1, t_2, ..., t_p, u(.)) \\ & + \int_0^t A^\alpha X(t-s) f\Big(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\Big) ds \end{aligned}$$

then

$$\begin{aligned} \| u(t) \|_\alpha \leq {} & M \| A^\alpha u_0 \| + M \| A^\alpha g(t_1, t_2, ..., t_p, u(.)) \| + \\ & + \| \int_0^t A^\alpha X(t-s) f\Big(s, u(s)), \int_0^s k(s, \tau, u(\tau))d\tau\Big) ds \| \end{aligned}$$

$$\leq M[\,\|\, A^{\alpha}u_0 \,\| + \|\, A^{\alpha}g(t_1, t_2, ..., t_p, u(.)) \,\|\,] + q(t)a^{*}$$
$$+ q(t)\int_0^t (t-s)^{-\alpha} \,\|\, u(s) \,\|_{\alpha}\, ds.$$

By Gronwall's inequality, it follows that $\| u(t) \|_{\alpha} \leq C$ on $[0, a)$. □

Acknowledgements. The second author is thankful to the Fulbright Foundation for providing a travel grant to visit The University of Tennessee at Chattanooga and to Professor J. P. Dauer for providing a hospitable stay while at UTC during June and July 1996.

References

1. K. Balachandran and S. Ilamaran, *Existence and uniqueness of mild and strong solutions of a semilinear evolution equation with nonlocal conditions*, Indian J. Pure Appl. Math. **25** (1994), 411–418.
2. L. Byszewski, *Strong maximum principles for parabolic nonlinear problems with nonlocal inequalities together with integrals*, J. Appl. Math. Stoch. Anal. **3** (1990), 65–79.
3. L. Byszewski, *Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation* $u_{xt} = F(t, x, u, u_x)$, J. Appl. Math. Stoch. Anal. **3** (1990), 163–168.
4. L. Byszewski, *Theorem about the existence and uniqueness of continuous solution of nonlocal problem for nonlinear hyperbolic equation*, Appl. Anal. **40** (1991), 173–180.
5. L. Byszewski, *Theorem about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991), 494–506.
6. L. Byszewski, *Strong maximum principles for parabolic nonlinear problems with nonlocal inequalities together with arbitrary functions*, J. Math. Anal. Appl. **156** (1991), 457–470.

7. L. Byszewski, *Existence of approximate solution to abstract nonlocal Cauchy problem*, J. Appl. Math. Stoch. Anal. **5** (1992), 363–374.

8. L. Byszewski, *Uniqueness criterion for solution to abstract nonlocal Cauchy problem*, J. Appl. Math. Stoch. Anal. **162** (1991), 49–54.

9. C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.

10. G. Gripenberg, S. O. London, and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.

11. D. Jackson, *Existence and uniqueness of solutions to semilinear nonlocal parabolic equations*, J. Math. Anal. Appl. **172** (1993), 256–265.

12. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.