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ALMOST PERIODICITY OF SOME SOLUTIONS TO TO LINEAR ABSTRACT EQUATIONS

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Introduction

Let $E = E(\tau)$ be a Fréchet space where its topology τ is generated by a family of continuous semi-norms $Q = \{p_i\}_{i=1}^{\infty}$. Consider the following abstract evolution equation in E:

(1) $x'(t) = Ax(t), \; t \in \mathbb{R}$

with the following assumptions:

(H_1) A is a closed linear (generally unbounded) operator with domain $D(A)$ dense in E

(H_2) A is the infinitesimal generator of a family of continuous one-parameter group $T(t), \; t \in \mathbb{R}$

(H_3) $\forall \, p_i \in Q \;\; \exists \, p_j \in Q \;\; p_i(T(t)u) \le p_j(u)$
$\forall \, t \in \mathbb{R}, \forall \, u \in E$.

We proved in [4] that if E is a perfect Fréchet space, then every solution $x(t)$ of (1) such that $\{x'(t): t \in \mathbb{R}\}$ is relatively compact in E is almost periodic in E. In section 1 of this paper we drop the assumption on E which seems severe and replace it by a more simple condition on the operator A for a more general situation, to get the same conclusion of almost-periodicity of $x(t)$.

In section 2, we prove almost periodicity of solutions with relatively compact range in a Hilbert space. This result is known for the case A is symmetric (see [1] or [3] for a more general situation).

I. <u>Theorem</u> 1: Consider equation (1) in a (not necessarily perfect) Fréchet space E. Let A such as in (H_1)–(H_3) and with range $R(A)$ dense in E. Then every solution $x(t)$ of (1) with $x(0) = x_0 \in D(A)$ and $\{x'(t): t \in \mathbb{R}\}$ relatively compact in E, is almost–periodic.

<u>Proof</u>: We do it in 3 steps:

<u>Lemma</u> 1: Every solution of (1) has almost periodic derivative $x'(t)$.

<u>Proof</u>: Let $(\tau_n)_{n=1}^{\infty}$, be any given real sequence; then we can substract a subsequence $(s_n)_{n=i}^{\infty}$ such that $(x'(s_n))_{n=i}^{\infty}$ is a Cauchy sequence in E for $\{x'(t): t \in \mathbb{R}\}$ is relatively compact in E. But we have

$$\begin{aligned} x'(t + s_n) &= Ax(t + s_n) = AT(t + s_n)\, x(0) \\ &= AT(t)T(s_n)x(0) = AT(t)x(s_n) \\ &= T(t)Ax(s_n) = T(t)x'(s_n) \end{aligned}$$

therefore, for any n,m, we get

$$x'(t + s_n) - x'(t + s_m) = T(t)\,(x'(s_n) - x'(s_m))\,.$$

Given $p_i \in Q$, we can find $p_j \in Q$ such that

$$p_i(x'(t + s_n) - x'(t + s_m)) \le p_j(x'(s_n) - x'(s_m))$$

for every $t \in \mathbb{R}$. We conclude using Bochner's criteria.

<u>Lemma</u> 2: Every solution $x(t)$ of (1) with $x(0) \in R(A)$ is almost periodic.

<u>Proof</u>: $x(0) \in R(A)$ implies existence of an element $u \in D(A)$ such that $Au = x(0)$. We have

$$x(t) = T(t)x(0) = T(t)Au = AT(t)u = Av(t) = v'(t)$$

where $v(t) = T(t)u$ is a solution of (1) with $v(0) = u$. Therefore $v'(t)$ and consequently $x(t)$ is almost periodic by lemma 1.

<u>Proof of the theorem</u>: Let $x(t)$ be a solution of (1) with $x(0) = x_0 \in D(A)$. As $R(A)$ is dense in E we can find a sequence $(b_n)_{n=1}^{\infty}$ in $R(A)$ such that $\lim_{n\to\infty} b_n = x_0$. Take $(y_n(t))_{n=1}^{\infty}$ as solutions of (1) with $y_n(0) = b_n$, $\forall n$. By Lemma 2, every $y_n(t)$ is almost periodic. Now it is clear that $(y_n(t))_{n=1}^{\infty}$ converge to $x(t)$, uniformly on $\mathbb{R}$.

In fact, $x(t) - y_n(t) = T(t)x_0 - T(t)b_n = T(t)(x_0 - b_n)$. Therefore for any $p_i \in Q$, there exists $p_j \in Q$ such that

$$p_i(x(t) - y_n(t)) \le p_j(x_0 - b_n),$$

for every $t \in \mathbb{R}$. The Theorem is proved.

II. <u>Theorem</u> 2: Consider equation (1) in a real Hilbert space H with assumptions (H_1) and (H_2). Then any solution $x(t) = T(t)x_0$ such that

(H_4) $x_0 \in Ker(A + A^*)$, A^* is the adjoint of A.

(H_5) $\{x(t): t \in \mathbb{R}\}$ is relatively compact in H is almost–periodic.

<u>Proof</u>: Clearly $x(t) \in Ker(A + A^*)$, $\forall t \in \mathbb{R}$. Otherwise there exists $t_0 \in \mathbb{R}$ such that $x(t_0) = T(t_0)x_0 \notin Ker(A + A^*)$ i.e. $\theta \neq (A + A^*)u(t_0) = (A + A^*)T(t_0)x_0$; this implies

$$A^*T(t_0)x_0 \neq -AT(t_0)x_0$$

or $T(t_0)(A + A^*)x_0 \neq \phi$, which is false for $(A + A^*)x_0 = \theta$. Now consider the continuous function $\phi: \mathbb{R} \to \mathbb{R}^+$ defined by $\phi(t) = \|x(t)\|^2$. Then it is constant on $\mathbb{R}$ for $\phi'(t) = (Ax(t), x(t)) + (x(t), Ax(t))$

$$= ((A + A^*)x(t), x(t)) = 0, \forall t \in \mathbb{R}$$

as shown above. We have $\phi(t) = \phi(0)$, $\forall t \in \mathbb{R}$ i.e. $\|x(t)\| = \|x(0)\|, \forall t \in \mathbb{R}$. Let $y(t) = x(t + s)$, defined on $\mathbb{R}$ for s fixed. Then $y_s'(t) = x'(t + s) = Ax(t + s) = Ay_s(t)$ which means $y_s(t) = y_s(0), \forall t \in \mathbb{R}$. Take s_1 and s_2 fixed in $\mathbb{R}$. Then

$$(y_{s_1}(t) - y_{s_2}(t))' = A(y_{s_1}(t) - y_{s_2}(t)) \, , \forall t \in \mathbb{R} \, .$$

Therefore

$$\|y_{s_1}(t) - y_{s_2}(t)\| = \|y_{s_1}(0) - y_{s_2}(0)\|, \forall t \in \mathbb{R}$$

i.e. $\|x(t + s_1) - x(t + s_2)\| = \|x(s_1) - x(s_2)\|, \forall t \in \mathbb{R}$.

Let $(s_n)_{n=1}^{\infty}$ be an arbitrary real sequence. Then we can substract a subsequence which we call again $(s_n)_{n=1}^{\infty}$ such that $(x(s_n))_{n=1}^{\infty}$ is a Cauchy sequence in H , for $x(t)$ has a relatively compact range in H .

Now

$$\sup_{t \in \mathbb{R}} \|x(t + s_n) - x(t + s_m)\| = \|x(s_n) - x(s_m)\|$$

The conclusion follows by Bochner's criteria.

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REFERENCES

[1] AMERIO, L. and PROUSE, G., Almost–Periodic Functions and Functional Equations, Van Nostrand Reinhold Co., 1971.

[2] CORDUNEANU, C., Almost–Periodic Functions, Interscience Publishers, John Wiley and Sons, New York, 1968.

[3] N'GUÉRÉKATA, G.M., Remarques sur les solutions presque–périodiques de l'éqation $(d/dt - A)\, x = 0$, Can. Math. Bull. vol. 25 (1), (1982) 121–123.

[4] N'GUÉRÉKATA, G.M., Notes on almost periodicity in topological vector spaces, Int. J. Math. & Math. Sci. vol. 9 No. 1, (1986) 201–204.

[5] ROBERTSON, A.P. and ROBERTSON, W., Topological Vector Spaces, Cambridge University Press, 1973.