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Classroom Notes

A SIMPLE PROOF OF ZORN'S LEMMA

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In this paper we give a simple and natural proof of Hausdorff's principle of maximality; then by the usual argument we obtain Zorn's lemma.

Theorem 1 (Hausdorff's principle of maximality). Let A be a nonvoid (partial) ordered set, with the order denoted by " $\leq$ ". There is a subset $B \subsetneq A$, maximal with respect to the inclusion in the class of total ordered subsets of A.

Proof. Let $\mathcal{A}$ be the class of total ordered subsets of A, i.e.:

$$\mathcal{A} = \{B|\, B \subseteq A;\ B \neq \emptyset;\ \forall x, y \in B \Rightarrow x \leq y \text{ or } y \leq x\}\ (\subseteq \mathcal{P}(A)).$$

Obviously, for every $x \in A$ we have $\{x\} \in \mathcal{A}$.

Let us assume by contradiction that in $\mathcal{A}$ there are not maximal elements with respect to the inclusion. It follows that for every $B \in \mathcal{A}$ we have

$$\emptyset \neq \{x|x \in A\backslash B,\ B \cup \{x\} \in \mathcal{A}\} = B'(\in \mathcal{P}(A)).$$

Thus we obtain the function $\mathcal{F} : \mathcal{A} \to \mathcal{P}(A)$, defined by

$$\mathcal{F}(B) = B',\ \forall B \in \mathcal{A}$$

According to the axiom of selection, there is a selection function $F : \mathcal{A} \to A$ so that $F(B) \in B' = \mathcal{F}(B),\ \forall B \in \mathcal{A}$.

We define the function $f : \mathcal{A} \to \mathcal{A}$ by

$$f(B) = B \cup \{F(B)\},\ \forall B \in \mathcal{A}.$$

Let $x_0 \in A$ be an arbitrary fixed element. We consider the set

$$\mathcal{A}_0 = \{B|B \in \mathcal{A};\ x_0 \in B\}.$$

Because $\{x_0\} \in \mathcal{A}_0$, it follows that $\mathcal{A}_0 \neq \emptyset$. Because $B \subsetneq f(B),\ \forall B \in \mathcal{A}$, it results that $f(B) \in \mathcal{A}_0,\ \forall B \in \mathcal{A}_0$.

We denote

$$\widetilde{\mathcal{B}} = \{\mathcal{B} | \mathcal{B} \subseteq \mathcal{A}_0 : \{x_0\} \in \mathcal{B}; \forall B \in \mathcal{B} \Rightarrow f(B) \in \mathcal{B};$$
$$(\forall \mathcal{C} \subseteq \mathcal{B}, \mathcal{C} \neq \emptyset, (\forall C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \subseteq C_2 \text{ or } C_2 \subseteq C_1)) \Rightarrow \bigcup_{C \in \mathcal{C}} C \in \mathcal{B}\}.$$

Let $\mathcal{C} \subseteq \mathcal{A}_0$, $\mathcal{C} \neq \emptyset$, so that for every $C_1, C_2 \in \mathcal{C}$ we have $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. Let $x, y \in \bigcup_{C \in \mathcal{C}} C$. It results that there are $C_1, C_2 \in \mathcal{C}$ so that $x \in C_1$ and $y \in C_2$. It follows that $x, y \in C_1$ or $x, y \in C_2$. Because $C_1, C_2 \in \mathcal{A}$ it results that $x \leq y$ or $y \leq x$. Hence $\bigcup_{C \in \mathcal{C}} C \in \mathcal{A}_0$. It follows that $\mathcal{A}_0 \in \widetilde{\mathcal{B}}$, hence $\widetilde{\mathcal{B}} \neq \emptyset$.

We denote

$$\bigcap_{\mathcal{B} \in \widetilde{\mathcal{B}}} \mathcal{B} = \mathcal{B}_0.$$

Obviously, $\mathcal{B}_0 \in \widetilde{\mathcal{B}}$ and $\mathcal{B}_0 \in \mathcal{B}, \forall \mathcal{B} \in \widetilde{\mathcal{B}}$.

We show that $\mathcal{B}_0$ is a total ordered set. Let

$$\mathcal{C}_0 = \{C | C \in \mathcal{B}_0; \forall B \in \mathcal{B} \Rightarrow B \subseteq C \text{ or } C \subseteq B\} \ (\subseteq \mathcal{B}_0).$$

Because $\{x_0\} \subseteq B, \ \forall B \in \mathcal{B}_0$, it follows that $\{x_0\} \in \mathcal{C}_0$, hence $\mathcal{C}_0 \neq \emptyset$. Obviously, $\mathcal{C}_0$ is a total ordered set.. We prove that $\mathcal{B}_0 = \mathcal{C}_0$. It is sufficient to show that $\mathcal{C}_0 \in \widetilde{\mathcal{B}}$.

Let $\mathcal{D} \subseteq \mathcal{C}_0$. Because $\mathcal{C}_0$ is a total ordered set, it follows that $\mathcal{D}$ is a total orderd set. It results that $\bigcup_{D \in \mathcal{D}} D \in \mathcal{B}_0$. Let $B \in \mathcal{B}_0$. For every $D \in \mathcal{D}$ we have $D \subseteq B$ or $B \subseteq D$. If $D \subseteq B, \forall D \in \mathcal{D}$, then $\bigcup_{D \in \mathcal{D}} D \subseteq B$. If there is $D \in \mathcal{D}$ so that D is not a subset of B, then $B \subseteq D$; it follows that $B \subseteq \bigcup_{D \in \mathcal{D}} D$. Thus, for every $B \in \mathcal{B}_0$ we have $B \subseteq \bigcup_{D \in \mathcal{D}} D$ or $\bigcup_{D \in \mathcal{D}} D \subseteq B$. Hence $\bigcup_{D \in \mathcal{D}} D \in \mathcal{C}_0$.

Let $C \in \mathcal{C}_0$. We show that $f(C) \in \mathcal{C}_0$. Obviously, for every $B \in \mathcal{B}_0$, if $C \subseteq B \subseteq f(C)$ then $B = C$ or $B = f(C)$. So, if $f(C) \in \mathcal{C}_0$ then $\mathcal{B}_0 = \mathcal{B}_C$, where we denoted

$$\mathcal{B}_C = \{B | B \in \mathcal{B}_0; B \subseteq C \text{ or } f(C) \subseteq B\} \ (\subseteq \mathcal{B}_0).$$

Conversely, if $\mathcal{B}_0 = \mathcal{B}_C$ then obviously we have $f(C) \in \mathcal{C}_0$. Hence $f(C) \in \mathcal{C}_0 \Leftrightarrow \mathcal{B}_0 = \mathcal{B}_C$. We show that $\mathcal{B}_0 = \mathcal{B}_C$. It is sufficient to show that $\mathcal{B}_C \in \widetilde{\mathcal{B}}$.

Because $\{x_0\} \subseteq C \ (\in \mathcal{C}_0 \subseteq \mathcal{B}_0)$ it follows that $\{x_0\} \in \mathcal{B}_C$.

Let $B \in \mathcal{B}_C$. We have $B \subseteq C$ or $f(C) \subseteq B$. Because $C \in \mathcal{C}_0$ and $f(B) \in \mathcal{B}_0$ we have $f(B) \subseteq C$ or $C \subseteq f(B)$. Consequently, we discriminate the following four cases:

1). $B \subseteq C$; $f(B) \subseteq C$. We have $f(B) \subseteq C$.

2). $B \subseteq C$; $C \subseteq f(B)$. We denote $B \subseteq C \subseteq f(B) = B \cup \{F(B)\}$. It follows that $C = B$ or $C = f(B)$. Hence $f(C) \subseteq f(B)$ or $f(B) \subseteq C$.

3). $f(C) \subseteq B$; $f(B) \subseteq C$. It follows that $C \subseteq f(C) \subseteq B \subseteq f(B) \subseteq C$, hence $C = f(C)$, absurd. It results this case is impossible.

4). $f(C) \subseteq B$; $C \subseteq f(B)$. We have $f(C) \subseteq B \subseteq f(B)$, hence $f(C) \subseteq f(B)$. In conclusion, for every $B \in \mathcal{B}_C$ we have $f(B) \in \mathcal{B}_C$.

Let $\mathcal{E} \subseteq \mathcal{B}_C$ be so that $\forall B_1, B_2 \in \mathcal{E}$ we have $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Because $\mathcal{B}_C \subseteq \mathcal{B}_0$ it results that $\bigcup_{B \in \mathcal{E}} B \in \mathcal{B}_0$. If $B \subseteq C, \forall B \in \mathcal{E}$, then $\bigcup_{B \in \mathcal{E}} B \subseteq C$. If there is $B \in \mathcal{E}$ so that B is not a subset of C, it follows that $f(C) \subseteq B$. It results that $f(C) \subseteq \bigcup_{B \in \mathcal{E}} B$. Hence, in both cases we have $\bigcup_{B \in \mathcal{E}} B \in \mathcal{B}_C$.

From the considerations above it follows that $\mathcal{B}_C \in \widetilde{\mathcal{B}}$. It results that $\mathcal{B}_0 \subseteq \mathcal{B}_C (\subseteq \mathcal{B}_0)$, hence $\mathcal{B}_0 = \mathcal{B}_C$. It follows that $f(C) \in \mathcal{C}_0$.

From the considerations above it follows that $\mathcal{C}_0 \in \widetilde{\mathcal{B}}$. It results that $\mathcal{B}_0 \subseteq \mathcal{C}_0 (\subseteq \mathcal{B}_0)$. Hence $\mathcal{B}_0 = \mathcal{C}_0$. The set $\mathcal{C}_0$ being total ordered, it follows that $\mathcal{B}_0$ is a total ordered set. Consequently, $\bigcup_{B \in \mathcal{B}_0} B = B_0 \in \mathcal{B}_0$ and $f(B_0) \in \mathcal{B}_0$. It results that $f(B_0) \subseteq B_0$, absurd, because $B_0 \subseteq f(B_0)$ and $B_0 \neq f(B_0)$. It follows that the assumption that in $\mathcal{A}$ there is no maximal element is false, and the proof is achieved.

Theorem 2 (Zorn's Lemma). Let A be a nonvoid inductive ordered set, with the order denoted by " $\leq$ ", i.e.: $(\forall B \subseteq A, \forall x, y \in B \Rightarrow x \leq y$ or $y \leq x) \Rightarrow \exists m_B \in A$ so that $C \leq m_B$, $\forall C \in B$. For every $a_0 \in A$ there is $m \in A$ maximal so that $a_0 \leq m$.

Proof. Let $a_0 \in A$ be an arbitrary fixed element. Let $A_0 = \{a \in A | a_0 \leq a\}$. According to Theorem 1, there is $B_0 \subseteq A_0$ total ordered and maximal with respect to the inclusion. According to the hypothesis there is $m \in A$ so that $a_0 \leq C \leq m$, $\forall C \in B_0$. The element m is a maximal element, because if there is $m' \in A$ so that $m < m'$, then $B_0 \cup \{m'\}$ is total ordered, contradicting the maximality of B_0.

Remark 1. Let A be a partial ordered set. The set of the total ordered parts of A is inductive ordered with respect to the inclusion. According to Theorem 2, there is a maximal total ordered part. Hence, Theorem 1 is a consequence of Theorem 2.

Remark 2. From the proof of Theorem 2 and from the Remark 1, it follows that Theorem 1 and Theorem 2 are equivalent.

Remark 3. The proof of Theorem 1 is a consistent simplification of the proof of Hausdorff's principle of maximality via the Lemma of Teichmüler-Tuckey (see [1]). Thus, the natural place of the Lemma of Teichmüler-Tuckey in the theory is a corollary of Zorn's Lemma, equivalent with this one and also with the principle of maximality of Hausdorff.

REFERENCES

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