

ON THE CONSTRUCTIBILITY WITH RULER AND COMPASS
OF A MINIMUM CHORD IN A PARABOLA

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ABSTRACT. In this note we give a constructive description of all the points inside a parabola admitting a minimum chord of fixed direction.

It is a standard result in synthetic geometry that the minimum chord through a given point inside a circular region, i.e., the shortest segment joining two points on the circle and passing through the given point, is precisely the chord perpendicular on the diameter containing the point, if the point is not the center of the circle. For the center itself, any diameter is a chord of minimum length. For more general convex regions the problem of constructibility, only with an unmarked ruler and a compass, of a minimum chord is unsolvable (see below). In such cases all one can do is *approximate* a minimum chord with the help of a computer (cf. [DO], for polygonal convex regions). However, the related problem of constructing all the points inside a smooth convex region which admit a minimum chord of a given direction, *may be* solvable. It is the purpose of this note to outline such a construction for conical regions. For simplicity, we consider here only the case of parabolas. We find that all the points inside a given parabola admitting a minimum chord of given direction form a straight segment, whose endpoints are constructible with unmarked ruler and compass. The arguments used in proving this claim involve elementary calculus and analytic geometry. Of course, it would be of interest to explain the construction solely within the framework of synthetic geometry, just as for circular regions.

Let \mathcal{P} be a parabola with focus F and directrix d in an Euclidean plane. In a coordinate system XY where F has coordinates $(0, c)$, $c > 0$, and d has equation $Y = -c$, the parabola \mathcal{P} is given by the equation $X^2 = 4cY$. Let P be a point inside (or on) \mathcal{P} , i.e., if P has coordinates (x, y) then $x^2 \leq 4cy$. Any direction, other than the axis of the parabola, $X = 0$, is uniquely determined by a real number m (its slope), and the chord through P having direction m is the straight segment whose endpoints are the two intersection points of the parabola \mathcal{P} with the line l passing through P , with slope m . Analytically, if $P_i(x_i, y_i)$, $i = 1, 2$, are the endpoints of this chord, then

$$x_{1,2} = 2\sqrt{c}(\sqrt{cm} \pm \sqrt{cm^2 - xm + y}), \quad y_{1,2} = y + (x_{1,2} - x)m.$$

Thus, the square of the length of this chord is

$$\overline{P_1P_2}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 - x_2)^2(m^2 + 1) = 16c(cm^2 - xm + y)(m^2 + 1).$$

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If (x, y) is fixed and m is variable the chord P_1P_2 will have minimum length if and only if its slope minimizes the polynomial function

$$f(m) := (cm^2 - xm + y)(m^2 + 1) = cm^4 - xm^3 + (c + y)m^2 - xm + y.$$

Obviously, a minimum chord through $P(x, y)$ exists, and its slope is a critical point of f , i.e., a solution of the cubic equation in m ,

$$(1) \quad 4cm^3 - 3xm^2 + 2(c + y)m - x = 0.$$

Depending on (x, y) , the above equation can have one or three real roots, counted with multiplicities. Let us classify the points (x, y) according to the number of real roots of Equation 1. The transformation $m = t + x/4c$ shows that this number equals the number of real roots of the polynomial equation in t ,

$$t^3 + pt + q = 0, \quad p := -\frac{3}{16c^2}x^2 + \frac{1}{2c}y + \frac{1}{2}, \quad q := -\frac{1}{32c^3}x^3 + \frac{1}{8c^2}xy - \frac{1}{8c}x$$

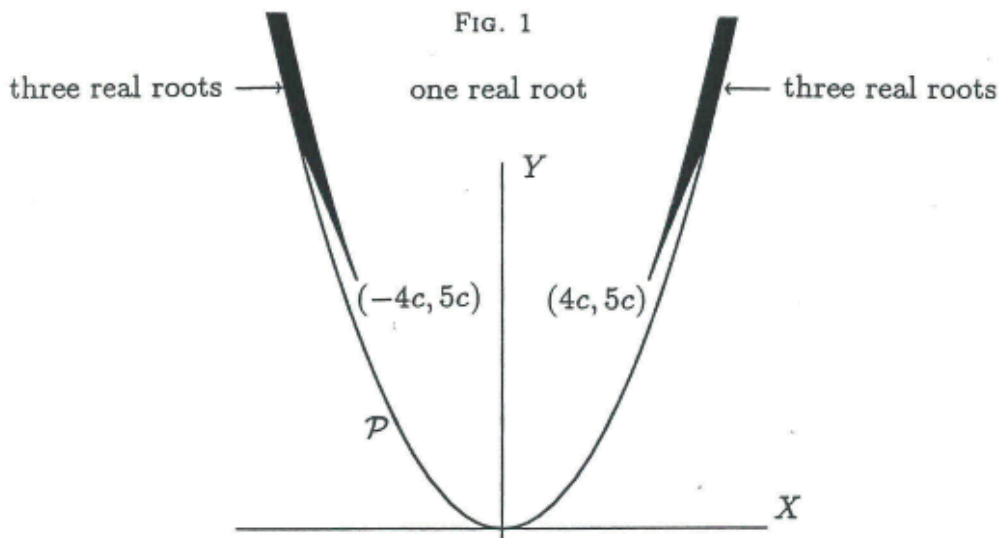
Now the equation $t^3 + pt + q = 0$ has three real roots if $p = q = 0$ or $p < 0, 4p^3 + 27q^2 \leq 0$, and only one otherwise. $p = q = 0$ gives $(x, y) = (\pm 4c, 5c)$ and $p < 0, 4p^3 + 27q^2 \leq 0$ is equivalent to the system of inequalities

$$\frac{1}{4c}x^2 \leq y < \frac{3}{8c}x^2 - c, \quad 27x^4 - 9(y^2 + 14cy + c^2)x^2 + 32c(y + c)^3 \leq 0.$$

or

$$\begin{aligned} \frac{1}{4c}x^2 &\leq y < \frac{3}{8c}x^2 - c, \quad y \geq 5c, \\ \frac{1}{6}y^2 + \frac{7c}{3}y + \frac{c^2}{6} - \frac{y-5c}{18}\sqrt{3(3y+c)(y-5c)} &\leq x^2 \\ x^2 &\leq \frac{1}{6}y^2 + \frac{7c}{3}y + \frac{c^2}{6} + \frac{y-5c}{18}\sqrt{3(3y+c)(y-5c)} \end{aligned}$$

Fig. 1 shows a typical distribution of the number of real roots of Equation 1, according to the position of the point (x, y) inside the parabola \mathcal{P} .



When there is only one distinct real root, it represents with certainty the slope of the minimum chord; when there are two distinct real roots, the simple one corresponds to the minimum chord, and when there are three distinct real roots the slope of the minimum chord is either the smallest or the largest of the roots, or both. At any rate, knowing the number of real roots does not help in constructing, with unmarked ruler and compass, the minimum chord. Indeed, according to the fundamental theorem of constructibility with unmarked ruler and compass, [S, Ch. 5], if one denotes by k the subfield of \mathbb{R} generated by c, x, y , the degree $[k(\theta) : k]$ of the field extension $k(\theta)$ of k , obtained by adjoining to k a real root θ of Equation 1, must be a power of 2, in order for θ (and thus the minimum chord) to be constructible. Or, it is easy to find (rational) numbers c, x, y such that this degree is 3, e.g., $c = x = y = 1$. We want to stress here that we look for a geometric way to construct the minimum chord *exactly*. Otherwise, numerical methods can be used for any given (x, y) to approximate the solutions of Equation 1, and determine which one corresponds to a minimum chord by evaluating the function f at them.

Let us now fix a slope $s \geq 0$. From Equation 1, all points (x, y) which admit a minimum chord of slope s are on the line

$$(2) \quad -(3s^2 + 1)X + 2sY + 2cs(2s^2 + 1) = 0.$$

More precisely, they belong to the straight segment whose endpoints are obtained by intersecting the line given by Equation 2 with the parabola \mathcal{P} . If $s > 0$ their X -coordinates are $2cs$ and $2c(2s^2 + 1)/s$. If $s = 0$ they belong to the nonnegative part of the Y -axis. Alternatively, a necessary condition for a minimum chord through (x, y) to have slope $s, s > 0$, is

$$(3) \quad y = \frac{3s^2 + 1}{2s}x - c(2s^2 + 1), \quad 2cs \leq x \leq 2c\frac{2s^2 + 1}{s}.$$

Not all the points of this segment admit minimum chords with slope s , but only a subsegment of it. The following proposition characterizes this subsegment.

Proposition 4. *A point (x, y) inside the parabola \mathcal{P} admits a minimum chord with slope s if and only if*

- a) $x = 0$, for $s = 0$.
- b) $y = \frac{3s^2 + 1}{2s}x - c(2s^2 + 1)$, $2cs \leq x \leq 2c\frac{2s^2 + 1}{s}$, for $s > 0$.

Proof. Clearly a) is true, since if $x = 0$ Equation 1 has $m = 0$ as its only real solution.

b) If (x, y) satisfies Equation 3 for some $s > 0$, then

$$f(m) - f(s) = (m - s)^2 \left(cm^2 + (2cs - x)m + \left(cs^2 - \frac{sx}{2} + \frac{x}{2s} \right) \right).$$

Evidently $f(s)$ is a minimum value for $f(m)$ if and only if the discriminant of the quadratic polynomial in m ,

$$cm^2 + (2cs - x)m + cs^2 - \frac{sx}{2} + \frac{x}{2s}$$

is not positive, i.e., $x(x - 2cs - 2c/s) \leq 0$. This inequality, combined with the restriction on x given by Equation 3, proves b). \square

This analytic finding translates neatly into constructive geometry.

Corollary 5. Let \mathcal{P} be a parabola with focus F , vertex V , and directrix d . Let G be the intersection point of d with the axis of symmetry σ of \mathcal{P} . Let M be a point on \mathcal{P} , $M \neq V$. Through F draw the parallel line α to the straight segment GM . If the tangent line τ to \mathcal{P} at M intersects σ at T , construct the point S on σ such that $\overline{GS} = 2\overline{TV}$ and G is between V and S . Let N be the intersection of the line β , passing through M and S , with the line α .

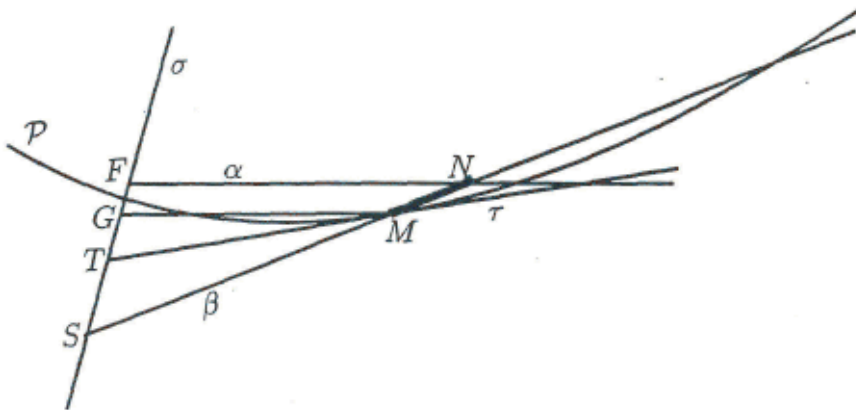
Then the straight segment MN is the geometric locus of all the points inside (or on) \mathcal{P} admitting a minimal chord parallel to the tangent line τ (see Fig. 2). If $M = V$, this geometric locus is the half part of the axis of symmetry σ inside \mathcal{P} .

Proof. Using analytic geometry in the coordinate system XY , we have $F = (0, c)$, $V = (0, 0)$, and $G = (0, -c)$. If the tangent line τ to \mathcal{P} at M has slope s , then $M = (2cs, cs^2)$. Due to the symmetry of \mathcal{P} , we can assume $s > 0$, if $M \neq V$. Thus the line α has slope $(s^2 + 1)/2s$ and β is given by the equation $Y + c(2s^2 + 1) = [(3s^2 + 1)/2s]X$. Since α is given by the equation $Y - c = [(s^2 + 1)/2s]X$, we see that $\alpha \cap \beta := N$ has coordinates $(2c(s^2 + 1)/s, c[(s^2 + 1)/2s]^2 + c)$. Thus the straight segment MN identifies with the portion of the line β ,

$$Y + c(2s^2 + 1) = \frac{3s^2 + 1}{2s}X, \quad X \text{ between } 2cs \text{ and } 2c\frac{s^2 + 1}{s}.$$

The corollary follows from Proposition 4. \square

FIG. 2



REFERENCES

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