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Univalence of certain integral operators

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Abstract

Many authors studyed the problem of integral operators which preserve the class of univalent functions. The results due to Kim and Merkes [2] and Pfaltzgraff [7] are well-known. A generalization of these results are given in paper [5]. In this note we shall study the analyticity and the univalence of some integral operators if the function f belongs to some special subclasses of univalent functions.

1 Introduction

Let A denote the class of functions f which are analytic in the unit disk $U = \{ z \in C : |z| < 1 \}$ with $f(0) = 0$ and $f'(0) = 1$.
Let S denote the class of functions $f \in A$, f univalent in U.

Theorem 1.1 *([2]). Let $f \in S$, $\beta \in C$. If $|\beta| \leq 1/4$, then the function F,*

$$F(z) = \int_0^z \left(\frac{f(u)}{u} \right)^\beta du$$

is univalent in U.

Theorem 1.2 *([7]). Let $f \in S$, $\delta \in C$. If $|\delta| \leq 1/4$, then the function F ,*

$$F(z) = \int_0^z (f'(u))^\delta du$$

is univalent in U.

Theorem 1.3 *([5]). Let $f \in S$, $n \in N$, α, β, $\delta \in C$. If $|\alpha - n| < n$, $|\beta| + |\delta| \leq (n - |\alpha - n|)/(4n)$, then the fuction*

$$F_{\alpha,\beta,\delta,n}(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{f(u^n)}{u^n} \right)^\beta (f'(u^n))^\delta du \right]^{1/\alpha}$$

is analytic and univalent in U.

The usual subclasses of the class S consisting of convex and starlike functions will be denoted by CV respectively S^*. Also we consider the subclasses of S defined as follows: (see for example [1])

$$S^*(\gamma,\varphi)=\left\{\ f\in S:\ \ Re\left(e^{i\varphi}\frac{zf'(z)}{f(z)}\right)>\gamma\cos\varphi,\quad z\in U\right\}$$

and

$$C(\gamma,\varphi)=\left\{\ f\in S:\ \ Re\left[e^{i\varphi}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>\gamma\cos\varphi,\quad z\in U\right\}\ ,$$

where $\varphi\in(-\pi/2,\pi/2)$, $\gamma\in[0,1)$.
We observe that $S^*=S^*(0,0)$ and $CV=C(0,0)$.

Theorem 1.4 *([4]). If $f\in S^*(\gamma,\varphi)$, $\beta\in C$, $|\beta|\leq\frac{1}{2(1-\gamma)\cos\varphi}$, then the function F ,*

$$F(z)=\int_0^z\left(\frac{f(u)}{u}\right)^\beta du$$

is univalent in U. The number $\frac{1}{2(1-\gamma)\cos\varphi}$ cannot be replace by any greater number.

Theorem 1.5 *([4]). If $f\in C(\gamma,\varphi)$, $\delta\in C$, $|\delta|\leq\frac{1}{2(1-\gamma)\cos\varphi}$, then the function F,*

$$F(z)=\int_0^z(f'(u))^\delta du$$

is univalent in U. The number $\frac{1}{2(1-\gamma)\cos\varphi}$ cannot be replace by any greater number .

2 Preliminaries

Lemma 2.1 *([3]). If $f\in S^*(\gamma,\varphi)$ and a is a fixed point from the unit disk U, then the function h,*

$$(1)\qquad h(z)=\frac{a\cdot z}{f(a)(z+a)(1+\bar{a}\cdot z)^\psi}\cdot f\left(\frac{z+a}{1+\bar{a}\cdot z}\right),\quad \text{where}$$

$$(2)\qquad \psi=e^{-2i\varphi}-2\gamma\cos\varphi e^{-i\varphi}$$

is a function of the class $S^*(\gamma,\varphi)$.

Theorem 2.1 *([6]). Let $f\in A$. Let α, β, c be complex numbers , $Re\alpha>0$, $Re(\alpha+2\beta)>0$, $Re\beta/\alpha>-1/2$, $|c(\alpha+\beta)+\beta|+|\beta|\leq|\alpha+\beta|$. If there exists an analytic function g, $g\in A$, such that*

$$|\ (1+c)\frac{f'(z)}{g'(z)}-1\ |<1\ ,\qquad (\forall z)\in U$$

$$\left|\left[(1+c)\frac{f'(z)}{g'(z)}-1\right]|z|^{2(\alpha+\beta)}+\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zg''(z)}{g'(z)}-\beta\right)\right|\leq 1$$

for all $z\in U\setminus\{0\}$*, then the function* F

$$F(z)=\left(\alpha\int_0^z u^{\alpha-1}f'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U*.*

The results obtained are proved by using Theorem 2.1 in the particular case $f\equiv g$ and $\beta=n-\alpha$, where $n\in N$. For this choise , from Theorem 2.1 we get the followig

Corollary 2.1 *Let* $g\in A$*. Let* α*,* c *be complex numbers and let* n *be a positive integer number. If* $|\alpha-n|<n$*,* $|c|<1$*,* $|cn+n-\alpha|+|n-\alpha|\leq n$ *and*

$$\left|c|z|^{2n}+\frac{1-|z|^{2n}}{n}\left(\frac{zg''(z)}{g'(z)}+\alpha-n\right)\right|\leq 1 \tag{3}$$

for all $z\in U$*, then the function*

$$G_\alpha(z)=\left(\alpha\int_0^z u^{\alpha-1}g'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U*.*

3 Main results

Theorem 3.1 *Let* $f\in S^*(\gamma,\varphi)$*,* α*,* $\beta\in C$ *and* $n\in N$*. If*

$$|\alpha-n|<|n| \quad \text{and} \quad |\beta|<\frac{n-|\alpha-n|}{2n(1-\gamma)\cos\varphi}\,, \tag{4}$$

then the function F*,*

$$F(z)=\left(\alpha\int_0^z u^{\alpha-1}\left(\frac{f(u^n)}{u^n}\right)^\beta du\right)^{1/\alpha} \tag{5}$$

is analytic and univalent in U*.*

Proof. Let $f\in S^*(\gamma,\varphi)$ and let h be the function defined by Lemma 2.1 , $h(z)=z+a_2z^2+\cdots$, $h\in S^*(\gamma,\varphi)$. From (1) we obtain

$$a_2=\frac{h''(0)}{2}=(1-|a|^2)\frac{f'(a)}{f(a)}-\frac{1+\psi|a|^2}{a}\,,$$

where ψ is given by (2). It follows that

$$\frac{af'(a)}{f(a)} = \frac{1 + a \cdot a_2 + \psi|a|^2}{1 - |a|^2}, \qquad (\forall) a \in U. \tag{6}$$

Also we know that for $f \in S^*(\gamma, \varphi)$, $f(z) = z + a_2 z^2 + \cdots$, we have(see for example [1])

$$|a_2| \leq 2(1-\gamma)\cos\varphi \tag{7}$$

Since the function f is univalent in U, we can choose the uniform branch of $\left(\frac{f(u^n)}{u^n}\right)^\beta$ equal to 1 at the origin, analytic in U and then the function g belongs to A, where

$$g(z) = \int_0^z \left(\frac{f(u^n)}{u^n}\right)^\beta du$$

We obtain

$$\frac{zg''(z)}{g'(z)} = \beta n \left(\frac{z^n f'(z^n)}{f(z^n)} - 1\right) \tag{8}$$

In view of (6) and (8), from (3) we get

$$\left| c|z|^{2n} + \frac{1 - |z|^{2n}}{n} \left(\frac{zg''(z)}{g'(z)} + \alpha - n\right)\right| = \tag{9}$$

$$= \left| \beta \cdot a_2 \cdot z^n + |z|^{2n} \left(c + \beta(\psi + 1) + \frac{n - \alpha}{n}\right) + \frac{\alpha - n}{n}\right|$$

If $c = -\beta(\psi + 1) + (\alpha - n)/n$ in view of (4) and (7) we have

$$|c| \leq |\beta| \cdot |\psi + 1| + \frac{|\alpha - n|}{n} = 2 \cdot |\beta|(1 - \gamma)\cos\varphi + \frac{|\alpha - n|}{n} < 1$$

and also

$$|cn + n - \alpha| + |n - \alpha| = |\beta| \cdot n \cdot |\psi + 1| + |n - \alpha| < n$$

Taking into account (7), the relation (9) becomes

$$\left| c|z|^{2n} + \frac{1 - |z|^{2n}}{n} \left(\frac{zg''(z)}{g'(z)} + \alpha - n\right)\right| \leq |\beta| \cdot |a_2| + \frac{|\alpha - n|}{n} < 1$$

From Corollary 2.1 we conclude that the function F defined by (5) is analytic and univalent in U.

Remark. For $\alpha = 1$ and $n = 1$ from Theorem 3.1 we obtain a restrictiv form of Theorem 1.4, with $|\beta| < \frac{1}{2(1-\gamma)\cos\varphi}$

Corollary 3.1 *Let* $f \in S^*$, α, $\beta \in C$ *and* $n \in N$. *If*

$$|\alpha - n| < n \qquad \text{and} \qquad |\beta| < \frac{n - |\alpha - n|}{2n}$$

then the function F *defined by (5) is analytic and univalent in* U.

Thus if we ask for f to be a starlike function we obtain for β a greater disk than that from Theorem 1.3 ($\delta = 0$).
For $\beta = \alpha - 1$ and $n = 1$ from Theorem 3.1 we get

Corollary 3.2 *Let* $f \in S^*(\gamma, \varphi)$, $\alpha \in C$. *If* $|\alpha - 1| < 1/(1 + 2(1 - \gamma)\cos\varphi)$ *then the function*

$$(10) \quad F(z) = \left(\alpha \int_0^z f^{\alpha - 1}(u) du\right)^{1/\alpha}$$

is analytic and univalent in U.

In particular if $f \in S^*$, $\alpha \in C$, $|\alpha - 1| < 1/3$, then the function F defined by (10) is analytic and univalent in U.

Example. *Let* α, $\beta \in C$, $n \in N$. *If*

$$|\alpha - n| < n \qquad \text{and} \qquad |\beta| < \frac{n - |\alpha - n|}{2n(1 - \gamma)\cos\varphi}$$

then the function

$$F(z) = z \cdot \left[\, G(2\beta(1 - \gamma)e^{-i\varphi}\cos\varphi,\ \frac{\alpha}{n},\ \frac{\alpha}{n} + 1;\ z^n) \,\right]^{1/\alpha}$$

is analytic and univalent in U, *where by* $G(a,\ b,\ c;\ z)$ *we noted the hypergeometric function.*
Proof. Let us consider the function f, $f \in S^*(\gamma, \varphi)$,

$$f(z) = z(1 - z)^{-2(1-\gamma)e^{-i\varphi}\cos\varphi}$$

Then, by Theorem 3.1 we obtain that the function F defined by (5) is analytic and univalent in U,

$$F(z) = \left(\alpha \int_0^z u^{\alpha - 1}(1 - u^n)^{-2\beta(1-\gamma)\cdot e^{-i\varphi}\cdot\cos\varphi} du\right)^{1/\alpha}$$

By the transformation $u = t^{1/n} \cdot z$ we have

$$F(z) = \left[\, \frac{\alpha}{n} z^\alpha \int_0^1 t^{\frac{\alpha}{n} - 1}(1 - tz^n)^{-2\beta(1-\gamma)\cdot e^{-i\varphi}\cdot\cos\varphi} dt \,\right]^{1/\alpha} =$$

$$= z \cdot \left[G(2\beta(1 - \gamma) \cdot e^{-i\varphi} \cdot \cos\varphi,\ \frac{\alpha}{n},\ \frac{\alpha}{n} + 1;\ z^n)\right]^{1/\alpha}$$

Theorem 3.2 *Let* $f \in C(\gamma,\varphi)$, α, $\delta \in C$ *and* $n \in N$. *If*

$$(11)\quad |\alpha - n| < n \qquad and \qquad |\delta| < \frac{n - |\alpha - n|}{2n(1-\gamma)\cos\varphi},$$

then the function F

$$(12)\quad F(z) = \left(\alpha \int_0^z u^{\alpha-1}\left(f'(u^n)\right)^\delta du\right)^{1/\alpha}$$

is analytic and univalent in U.

The proof of Theorem 3.2 is analogous to that of Theorem 3.1 and it uses the relationship between the classes $S^*(\gamma,\varphi)$ and $C(\gamma,\varphi)$: if $f \in C(\gamma,\varphi)$ then $g \in S^*(\gamma,\varphi)$, where $g(z) = zf'(z)$.
Thus if we ask for $f \in C(\gamma,\varphi)$ we obtain for δ a greater disk which assures the analyticity and the univalence of the function F defined by (12) than that from Theorem 1.3 ($\beta = 0$).

Remark. For $\alpha = 1$ and $n = 1$, from Theorem 3.2 we obtain a restrictiv form of Theorem 1.5 , with $|\delta| < \frac{1}{2(1-\gamma)\cos\varphi}$.

Corollary 3.3 *Let* $f \in CV$, α, $\delta \in C$ *and* $n \in N$. *If*

$$|\alpha - n| < n \qquad and \qquad |\delta| < \frac{n - |\alpha - n|}{2n}$$

then the function F *defined by (12) is analytic and univalent in* U.

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