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DIFFERENTIAL MEAN VALUE PROPERTIES FOR HALF-CONTINUOUS FUNCTIONS

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1. Introduction

Let X be a (real) locally convex space and $F : R \to X$, a function. Denote, for $r, s \in R, r < s$,

(1D1) $R_F(r,s) = F(s) - F(r), \quad Q_F(r,s) = \dfrac{R_F(r,s)}{s-r}$.

(These will be referred to as the *incrementary difference/quotient* of F on the interval $[r,s]$). The following statement involving such quantities is basic to considerations below. Take the points $a, b \in R, a < b$ in accordance with

(1H1) F is continuous over $[a,b]$,

and assume that, for a certain denumerable part A of $[a,b]$,

(1H2) F has a right derivative $d^{(+)}F(t)$, for $t \in]a,b[\setminus A$.

Let also B stand for a zero Lebesgue measure part of $]a,b[$ which includes A.

THEOREM 1. *Under these hypotheses,*

$$Q_F(a,b) \in \mathrm{clco}\{d^{(+)}F(t);\ t \in]a,b[\setminus B\}. \tag{1.1}$$

Moreover, if

(1H3) | $\mathrm{co}\{d^{(+)}F(t); t \in]a,b[\setminus B\}$ *has a nonempty interior in the minimal closed linear variety including it,*

then, (1.1) may be written in the stronger form

$$Q_F(a,b) \in \mathrm{co}\{d^{(+)}F(t); t \in]a,b[\setminus B\}. \tag{1.2}$$

A left counterpart of this result, based on (1H2) and (1H3) substituted by, respectively,

(1H2') F has a left derivative $d^{(-)}F(t)$, for $t \in]a,b[\setminus A$,

(1H3') | $\mathrm{co}\{d^{(+)}F(t); t \in]a,b[\setminus B\}$ has a nonempty interior in the minimal closed linear variety including it,

is directly obtainable from the above one, by simply reversing the (usual) ordering of the real axis. Note that both (1H2) and (1H2) hold under

(1H4) F has a derivative $dF(t)$, for $t \in]a,b[\setminus A$.

The basic relation (1.1)–usually referred to as an *equality mean value property* for F on the interval $[a,b]$– was obtained in 1964 by McLeod [8], under the lines in Aumann [1,ch.7,sect.2] .(It clearly includes the contribution in this area due to Sova [11]; see also Nashed [9,ch.1,sect.5] and the references therein). In fact, the used convention is especially related to the stronger version (1.2) of this relation, which may be appropriately written as

$$Q_F(a,b) = \sum_{i \in I} \lambda_i d^{(+)}F(t_i) \tag{1.3}$$

where I is a finite index set, $\{\lambda_i; i \in I\}$ are positive numbers with $\sum_{i \in I} \lambda_i = 1$ and $\{t_i; i \in I\}$, points in $]a,b[\setminus B$. For, if $X = R$ and (1H4) is accepted, then this is nothing but the Lagrange mean value theorem. So, further extensions of Theorem 1 – which, technically speaking, are naturally connected with (1H1) being removed– are not without interest. It is our main aim of this exposition to indicate such a device (under the lines in Averbukh and Smolyanov [2,ch.1,sect.3]) in a half-continuity setting; details will be given in Section 3. The preliminary material is exposed in Section 2. And, in Section 4, an application of these facts to inequality mean value properties is to be discussed.

2. Convex sets and cones in t.v.s.

Let X be a (real) topological vector space, the (linear) topology of it being induced by a certain family $\mathcal{V}$ of zero neighbourhoods. As usually, we denote by "int" and "cl" the topological *interior* and *closure* (of a set in X). Note, in this direction, that (cf. Precupanu [10,ch.1,sect.1])

$$\mathrm{cl}(M) = \cap\{M + V;\ V \in \mathcal{V}\},\quad M \subseteq X. \tag{2.1}$$

The subset A of X will be said to be *convex* in case

(2D1) $\lambda A + \mu A \subseteq (\lambda + \mu)A$,for all $\lambda, \mu \geq 0$.

Clearly,the class of all such sets is invariant to intersections ; hence,for each part M of X we may define its *convex cover* as

(2D2) $co(M) = \cap\{A \supseteq M; A = \text{convex}\}$.

Equivalently, $co(M)$ may be also defined as the class of all convex combinations

(2D3) $$x = \sum_{i \in I} \lambda_i x_i$$

where I is a finite index set, $\{\lambda_i; i \in I\}$ are positive numbers with $\sum_{i\in I} \lambda_i = 1$ and $\{x_i; i \in I\}$,points of M. In particular,when (for some $m \geq 1$)

(2H1) $X = R^m$ (endowed with its usual topology)

then we may arrange for card$(I) = m+1$ in the representation (2D3). And,if in addition,

(2H2) M has at most m connected components

this cardinality may be reduced to m ; see,for instance, Eggleston [6,ch.2,sect.3] .On the other hand,the class of all convex parts of X is invariant to the standard algebraic operations on X ; that is,

(2.2) $$P, Q = \text{convex} \Longrightarrow \alpha P + \beta Q \text{ is convex ,for } \alpha, \beta \in R.$$

Note also that for each convex part P of X containing the origin,

(2.3) $$\lambda P \subseteq \mu P, \quad \text{whenever } 0 \leq \lambda \leq \mu.$$

Concerning the behaviour of this class with respect to the topological operations on X ,denote,for each couple of points x, y in X

(2D4) $[x,y] = \{\lambda x + (1 - \lambda y); 0 \leq \lambda \leq 1\}$, $\quad]x,y[= [x,y] \setminus \{x,y\}$.

These will be referred to as the *closed* (respectively, *open*) *segment* determined by x and y. We have (cf. Bourbaki [4,ch.2,sect.2])

PROPOSITION 1. *Let A be a convex part of X .If the point x is interior to A and the point y is adherent to A then,the open segment $]x,y[$ is necessarily interior to A.*

This, in turn, yields the following answer to the posed question:

(2.4) $P = \text{convex} \Longrightarrow \text{int}(P), \text{cl}(P)$ are convex

(2.5) $P = \text{convex},\ \text{int}(P) \neq \emptyset \Longrightarrow \text{cl}(P) = \text{cl}(\text{int}(P)), \text{int}(P) = \text{int}(\text{cl}(P))$.

See the above references for details.

We call the part B of X, a *cone*, when

(2D5) $\lambda B \subseteq B, \lambda \geq 0; \quad B + B \subseteq B$.

As before, the class of all such sets is invariant to intersections; hence, for each part M of X we may introduce its *conical cover* as

(2D6) $\text{cone}(M) = \cap\{B \supseteq M; B = \text{cone}\}$.

Note that, cone(M) may be also defined as the class of all (conical) combinations (2D3), where I is a finite index set, $\{\lambda_i; i \in I\}$ are positive numbers and $\{x_i; i \in I\}$, points of M. In other words,

$$\text{cone}(M) = \cup\{\lambda \text{co}(M);\ \lambda \geq 0\}, \quad M \subseteq X. \tag{2.6}$$

Hence, the cardinality of the index set I is again reductible to $m + 1$ under (2H1) and, respectively, to m, under (2H2). On the other hand, the class of all cones in X is invariant to the usual algebraic operations on X, in the sense

$$P, Q = \text{cone} \Rightarrow \alpha P + \beta Q = \text{cone, for } \alpha, \beta \geq 0. \tag{2.7}$$

We also note the property (deduced from the above)

$$\text{cone}(M_1 \cup M_2) = \text{cone}(M_1) + \text{cone}(M_2), \quad M_1, M_2 \subseteq X. \tag{2.8}$$

Concerning the behaviour of this class with respect to the topological operations over X, it is to be remarked that

$$P = \text{cone} \Longrightarrow \text{cl}(P) = \text{cone}. \tag{2.9}$$

But a similar conclusion for the interior operation is not in general available, as simple examples show.

The following result will be useful for us. Denote, for each closed convex part M of X,

(2D7) $\text{asc}(M) = \cap\{\varepsilon(M - a);\ \varepsilon > 0\}$, for some $a \in M$.

The definition is consistent (it does not depend on the point $a \in M$ above);

and the resulting object is a cone (called the *asymptotic cone* of M). Further,call the cone N of X, *locally compact* if

(2D8) $N \cap V$ is compact (in N) for some $V \in \mathcal{V}$;

or,equivalently (cf. Bourbaki [op.cit.,ch.2,sect.7]) if

(2D8') N has a compact (in N) sole.

(Here,the part N_0 of N will be termed a *sole* ,provided

(2D9) $0 \notin N_0$ and $N = \cup\{\lambda N_0;\ \lambda \geq 0\}$.)

We now have

PROPOSITION 2. *Let the convex part A of X and the cone B of X be such that*

(2H3) $\mathrm{asc}(\mathrm{cl}(A)) \cap (-\mathrm{cl}(B))$ *is a linear space*

(2H4) *either* $\mathrm{asc}(\mathrm{cl}(A))$ *or* $\mathrm{cl}(B)$ *is locally compact.*

Then,necessarily,

$$\mathrm{cl}(A + B) = \mathrm{cl}(A) + \mathrm{cl}(B). \tag{2.10}$$

Proof. By the admitted facts and the Gwinner–Dedieu closedness criterion (see, for instance, Precupanu [op.cit.,ch.4,sect.4])

$$\mathrm{cl}(A) + \mathrm{cl}(B) \text{ is closed (hence, } \mathrm{cl}(A + B) = \mathrm{cl}(A) + \mathrm{cl}(B)).$$

On the other hand,by the representation formula (2.1),

$$\mathrm{cl}(A) + \mathrm{cl}(B) \subseteq \cap\{A + B + V + W;\ V, W \in \mathcal{V}\} = \mathrm{cl}(A + B).$$

This ends the argument. q.e.d.

A subset C of X is called a *closed linear variety* if

(2D11) C is the translate of a closed linear subspace (in X).

It is not hard to see that the class of all such sets is invariant to intersections. Hence,for each part M of X, we may define its *closed affine cover* as

(2D12) $\mathrm{aff}(M) = \cap\{C \supseteq M; C = \text{closed linear variety}\}$,

also referred to as the *minimal closed linear variety* that includes M. In particular,when $0 \in M$,then

$$\mathrm{aff}(M) = \text{the minimal closed linear subspace including } M.$$

Note that,whenever (2H1) is in use,one has

$$(\emptyset \neq)P = \text{convex} \Rightarrow \text{ the relative to aff}(P) \text{ interior of } P \text{ is nonempty.}$$

For a direct proof we refer to Eggleston [op.cit.,ch.1,sect.5].

3. Main results

We are now in position to make precise the considerations of the introductory part. Let X be a (real) locally convex space and $F : X \to R$, a function. For the arbitrary fixed $t \in R$, call the vector $x \in X$,a *right (left) derivative value* for F at t, when

$$\text{(3D1)} \quad \left| \begin{array}{l} Q_F(t, s_n) \to x \text{ ,for some } (s_n) \text{ with } s_n \downarrow t \\ (Q_F(r_n, t) \to x \text{ ,for some } (r_n) \text{ with } r_n \uparrow t). \end{array} \right.$$

The set of all such elements will be denoted $\mathcal{D}^{(+)}F(t)$ (and, respectively, $\mathcal{D}^{(-)}F(t)$). Note that

$$\text{(3.1)} \quad \left| \begin{array}{l} \mathcal{D}^{(+)}F(t) = \{d^{(+)}F(t)\} \; (\mathcal{D}^{(-)}F(t) = \{d^{(-)}F(t)\} \;) \\ \text{whenever the right (left) derivative exists.} \end{array} \right.$$

Further,call the vector $y \in X$,a *right (left) oscillation value* for F at t ,when

$$\text{(3D2)} \quad \left| \begin{array}{l} R_F(t, s_n) \to y \text{ ,for some } (s_n) \text{ with } s_n \downarrow t \\ (R_F(r_n, t) \to y \text{ ,for some } (r_n) \text{ with } r_n \uparrow t). \end{array} \right.$$

The set of all such elements will be denoted $\mathcal{E}^{(+)}F(t)$ (and, respectively, $\mathcal{E}^{(-)}F(t)$). Note that

$$\text{(3.2)} \quad \left| \begin{array}{l} \mathcal{E}^{(+)}F(t) = \{F(t+0) - f(t)\} \; (\mathcal{E}^{(-)}F(t) = \{F(t) - F(t-0)\} \;) \\ \text{whenever the right (left) limit exists.} \end{array} \right.$$

In particular, a relation like

(3D3) $\mathcal{E}^{(+)}F(t) = \{0\}$ (respectively, $\mathcal{E}^{(-)}F(t) = \{0\}$)

will be referred to as F being *continuous from the right (left)* at t.

Fix in the following $a, b \in R, a < b$, and assume that

(3H1) F is continuous from the left over $]a, b]$,

(3H2) $\mathcal{E}^{(+)}F(t)$ is nonempty. for $t \in [a, b[$.

Let also A be some denumerable part of $]a, b[$ with

(3H3) $\mathcal{D}^{(+)}F(t) \neq \emptyset$,for $t \in]a, b[\setminus A$.

Take a certain selection $t \vdash e^{(+)}F(t)$ of the multivalued map $t \vdash \mathcal{E}^{(+)}F(t)$. Note that (3H3) yields

$$0 \in \mathcal{E}^{(+)}F(t), \quad t \in]a, b[\setminus A. \tag{3.3}$$

So,an example of such a selection, denoted $t \vdash e_A{}^{(+)}F(t)$, is to be given as

$$\text{(3D4)} \quad \left| \begin{array}{l} e_A{}^{(+)}F(t) = 0, \quad t \in]a, b[\setminus A \ , \\ e_A{}^{(+)}F(t) \in \mathcal{E}^{(+)}F(t), \ \text{otherwise.} \end{array} \right.$$

Further, take a certain selection $t \vdash D^{(+)}F(t)$ of the multivalued map $t \vdash \mathcal{D}^{(+)}F(t)$. Finally,let B stand for a zero Lebesgue measure part of $]a, b[$ which includes A. Denote, for simplicity

(3D5) $Y(F) = \text{co}\{D^{(+)}F(t); t \in]a, b[\setminus B\}, Z(F) = \text{cone}\{e^{(+)}F(t); t \in [a, b[\}$.

THEOREM 2. *Under these hypotheses,*

$$Q_F(a, b) \in \text{cl}(Y(F) + Z(F)). \tag{3.4}$$

Hence,in particular,for each convex part Y of X and each cone Z of X with

(3H4) $D^{(+)}F(t) \in Y, \quad t \in]a, b[\setminus B$,

(3H5) $e^{(+)}F(t) \in Z, \quad t \in [a, b[$,

one has

$$Q_F(a, b) \in \text{cl}(Y + Z). \tag{3.5}$$

Before effectively developing the argument,we need the following facts to be found,e.g.,in Dieudonne [5,ch.8,sect.5] and Flett [7,ch.1,sect.10] :

PROPOSITION 3. *The following are valid:*

(i) for each denumerable part A of $]a,b[$ there exists an increasing function $h = h_A$ from R to itself, with

$$h(t+0) - h(t) > 0, \quad for \ t \in A.$$

(ii) for each zero Lebesgue measure part B of $]a,b[$ there exists an increasing comtinuous function $k = k_B$ from R to itself, with

$$d^{(+)}k(t) = \infty, \quad for \ t \in B.$$

Proof of Theorem 2. Without loss, assume

(3H6) the null element of X belongs to $Y(F)$.

(For,otherwise, take a certain point c in $]a,b[\setminus B$ and put

$$F_c(t) = F(t) - tD^{(+)}F(c), \quad t \in R.$$

Clearly, the general conditions (3H1)-(3H3) are holding for F_c. Moreover,in view of

$$Y(F_c) = Y(F) - D^{(+)}F(c), \quad Z(F_c) = Z(F),$$

it follows that (3H6) is fulfilled by this new function. And then,from (3.4) (written for F_c), we are done). Let $\eta > 0$ be arbitrary fixed. Define an increasing function $g_\eta : R \to R$ by

$$g_\eta(t) = t + \eta(h(t) + k(t)), \quad t \in R.$$

(Here, $h = h_A$ and $k = k_B$ are the functions introduced by Proposition 3). Let also V stand for a balanced convex neighbourhood of the origin. We introduce an ordering $\preceq$ over $[a,b]$ by the convention

(3D6) $t \preceq s$ iff $t \leq s$ and $F(s) - F(t) \in (g_\eta(s) - g_\eta(t))\text{cl}(Y(F) + Z(F) + V)$.

It is a simple matter to verify (by (3H1), (3H6), and the remarks in Section 2) that

| each ascending (modulo $\preceq$) sequence in $[a,b]$ is a
| Cauchy sequence bounded from above (modulo $\preceq$).

So,by the maximality principle in Turinici [12] there exists,for the starting point c in $]a,b[$,some maximal (modulo $\preceq$) point r in $[a,b]$ with $c \preceq r$. We intend to show that $r = b$. Suppose not (hence $r \in]a, b[$). For each t in $]r, b[$, a relation like $r \preceq t$ is impossible. So,necessarily,

$$R_F(r,t) \notin (g_\eta(t) - g_\eta(r))\mathrm{cl}(Y(F) + Z(F) + V), \quad r < t < b. \tag{3.7}$$

Now,three cases are open before us.

i) r is outside B. Let (t_n) be the sequence involved in the definition of $D^{(+)}F(r) \in Y(F)$. It follows by the very definition of this element that,a rank $n(V)$ may be found with

$$Q_F(r,t_n) \in Y(F) + V, \quad n \geq n(V).$$

Hence,for all such ranks n,

$$R_F(r,t_n) \in (t_n - r)\mathrm{cl}(Y(F) + V) \subseteq (g_\eta(t_n) - g_\eta(r))\mathrm{cl}(Y(F) + Z(F) + V),$$

which is contradictory to (3.7).

ii) r is in $B \setminus A$. Let again (t_n) be the sequence involved in the definition of $D^{(+)}F(r)$. It is clear,by Proposition 3 (the second part) that

$$Q_F(r,t_n)/Q_k(r,t_n) \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Hence,there must be some rank $n(V)$ with

$$Q_F(r,t_n)/Q_k(r,t_n) \in \eta V, \quad n \geq n(V).$$

But then,for all these ranks n,

$$R_F(r,t_n) \in \eta(k(t_n) - k(r))V \subseteq (g_\eta(t_n) - g_\eta(r))\mathrm{cl}(Y(F) + Z(F) + V),$$

again in contradiction to (3.7).

iii) r belongs to A. Let (t_n) be the sequence involved in the definition of $e^{(+)}F(r) \in Z(F)$. By Proposition 3 (the first part),

$$R_F(r,t_n)/R_h(r,t_n) \longrightarrow \frac{1}{h(t+0) - h(t)}\, e^{(+)}F(r) \in Z(F).$$

Hence, a rank $n(V)$ may be found with

$$R_F(r,t_n)/R_h(r,t_n) \in Z(F) + \eta V = \eta(Z(F) + V), \quad n \geq n(V).$$

Therefore, for all such ranks n,

$$R_F(r,t_n) \in \eta(h(t_n) - h(r))(Z(F) + V) \subseteq (g_\eta(t_n) - g_\eta(r))\mathrm{cl}(Y(F) + Z(F) + V),$$

which is contradictory with respect to (3.7).
Having explored all possibilities,our claim follows.But then, $c \preceq b$ gives

$$Q_F(t,b)/(1+\eta Q_{h+k}(c,b)) \in \mathrm{cl}(Y(F)+Z(F)+V), \quad \eta > 0.$$

So,passing to limit as $\eta \longrightarrow 0+$,

$$Q_F(c,b) \in \mathrm{cl}(Y(F)+Z(F)+V), \quad V \in \mathcal{V}.$$

And this,by the representation formula in the preceding section,yields

$$Q_F(c,b) \in \mathrm{cl}(Y(F)+Z(F)), \quad a < c < b;$$

wherefrom, for all such points c ,

$$R_F(c,b) \in (b-c)\mathrm{cl}(Y(F)+Z(F)) \subseteq (b-a)\mathrm{cl}(Y(F)+Z(F)). \tag{3.8}$$

Let (c_n) be the sequence involved in the definition of

$$e^{(+)}F(a) \in Z(F) = (b-a)Z(F).$$

Writing (3.8) for this sequence and then,passing to limit as $n \to \infty$, one gets

$$R_F(a,b) \in e^{(+)}F(a) + (b-a)\mathrm{cl}(Y(F)+Z(F)) \subseteq (b-a)\mathrm{cl}(Y(F)+Z(F)).$$

And,from this, (3.4) follows. The proof is thereby complete. q.e.d.

By convention, (3.4) is referred to as a *mean value property* for F over $[a,b]$. Note that,in general,the closure operator cannot be dropped in the second member of the underlying relation. This is shown from the following

Example. Let X be an infinite dimensional Banach space and Z , a cone in X with

(3H7) | there exists a sequence (v_n) in Z so that the series $\sum_n v_n$ converges and its sum, v, is outside Z.

(For example, Z may be taken as the nullspace of some discontinuous linear functional over X and

$$v_1 = x_1, \quad v_n = x_{n+1} - x_n, \quad n \geq 2,$$

where (x_n) is a sequence in Z converging to some point v in $X \setminus Z$). Let the function $F : R \to X$ be introduced as

$$\begin{aligned} F(t) &= 0 &&, t \le 0 \\ &= v_1 &&, 0 < t \le 1/2 \\ &= v_1 + v_2 &&, 1/2 < t \le 3/4 \\ &= \cdots &&, \cdots\cdots \\ &= v &&, t \ge 1. \end{aligned}$$

Take also $a = 0,\ b = 1$, and

$$A = \{1 - 2^{-n};\ \ n = 1, 2, ...\}, \quad B = A.$$

Clearly, (3H1) is fulfilled here. This is also true for (3H2)+(3H3),because

$$\left| \begin{array}{l} \mathcal{D}^{(+)}F(t) = \{0\} \text{ (hence } \mathcal{E}^{(+)}F(t) = \{0\}\text{), for } t \in]0, 1[\setminus A \\ \mathcal{E}^{(+)}F(1 - 2^{-n}) = v_{n+1}, \quad n = 0, 1, \ldots . \end{array} \right.$$

Hence, Theorem 2 applies to these data, in the form

$$v \in \text{clcone}\{v_1, v_2, ...\}.$$

But, in view of (3H7), it is clear that the closure operator cannot be dropped here; hence the claim.

This fact raises now the question of under which supplementary hypotheses is such a conclusion retainable. An appropriate answer may be given along the following lines. (The general assumptions (3H1)–(3H3) prevail).

THEOREM 3. *Suppose that*

(3H8) | *$Y(F) + Z(F)$ has a nonempty interior in the minimal closed linear variety including it.*

Then, the mean value property (3.4) *holds in the stronger form*

$$Q_F(a, b) \in Y(F) + Z(F). \tag{3.4'}$$

Hence, in particular, for each convex part Y of X and each cone Z of X fulfiling (3H4) *and* (3H5) *respectively, one has*

$$Q_F(a, b) \in Y + Z. \tag{3.5'}$$

Proof. Without loss,one may assume that (3H6)is in use.In this case,

$\mathrm{aff}(Y(F)+Z(F))$ is a closed subspace of X.

Now,again without any loss we may suppose

(3H9) $\mathrm{aff}(Y(F)+Z(F))=X$ (hence $\mathrm{int}(Y(F)+Z(F))\neq\emptyset$) ;

for,otherwise,we restrict all reasonings to the closed subspace $X_1=\mathrm{aff}(Y(F)+Z(F))$. We now claim that $Q_F(a,b)$ is interior to $Y(F)+Z(F)$. Suppose not; i.e., by combining with (3.4),

(3H10) $Q_F(a,b)$ is a boundary point of $Y(F)+Z(F)$.

By the classical Eidelheit's separation theorem (see, for instance Bourbaki [4,ch.2,sect.5]) there exists a continuous linear functional x^* and a real number λ, such that

$$x^*(x)\leq\lambda\leq x^*(Q_F(a,b)),\quad x\in Y(F)+Z(F). \tag{3.10}$$

As a direct consequence,one has,by a standard procedure,

$$\left| \begin{array}{ll} x^*(D^{(+)}F(t))\leq\lambda, & t\in]a,b[\setminus B \\ x^*(e^{(+)}F(t))\leq 0, & t\in[a,b[. \end{array}\right.$$

This tells us conditions (3H4)+(3H5) of Theorem 2 are fulfilled,with

$$Y=\{x\in X;\ x^*(x)\leq\lambda\},\quad Z=\{x\in X;\ x^*(x)\leq 0\}.$$

And then,by the conclusion of that result,

$$Q_F(r,s)\in \mathrm{cl}(Y+Z)=\mathrm{cl}(Y)=Y,\quad a\leq r<s\leq b;$$

or,in other words,

$$x^*(F(s)-F(r))\leq\lambda(s-r),\quad a\leq r<s\leq b.$$

We now claim each of these relations holds with equality.For,if a strict inequality is admitted over a subinterval $[r,s]$ of $[a,b]$,then

$$\left| \begin{array}{lll} x^*(F(b)-F(a)) & = & x^*(F(b)-F(s))+x^*(F(s)-F(r))+ \\ & & x^*(F(r)-F(a))<\lambda(b-a), \end{array}\right.$$

absurd by (3.10). Hence the claim. But then,

$$Y(F)+Z(F)\subseteq\{x\in X;\ x^*(x)=\lambda\}\ (\text{hence } \mathrm{int}(Y(F)+Z(F))=\emptyset).$$

This,however, contradicts (3H8) (in the form given by (3H9)).Therefore, (3H10) is false and the conclusion follows. q.e.d.

In particular,the regularity conditions (3H1)+(3H2) are fulfiled under (1H1).So,the couple of the results above comprises the (subsumed to Theorem 1) mean value McLeod's theorem [8] ;see also Averbukh and Smolyanov [2,ch.1,sect.3] .For a number of technical aspects we refer to the 1971 survey paper by Nashed [9,ch.1,sect.5].

It remains now to say what happens when (3H8) is no longer acceptable. In this direction,as a completion of the statement above,we have

THEOREM 4. *Suppose that*

(3H11) $\operatorname{asccl}(Y(F)) \cap (-\operatorname{cl}(Z(F)))$ *is a linear subspace*

(3H12) *either* $\operatorname{asccl}(Y(F))$ *or* $\operatorname{cl}(Z(F))$ *is locally compact*

Then,the mean value property (3.4) *holds in the simpler form*

$$Q_F(a,b) \in \operatorname{cl}(Y(F)) + \operatorname{cl}(Z(F)). \tag{3.4*}$$

Hence,in particular,for each convex part Y *of* X *and each cone* Z *of* X *fulfiling* (3H4) *and* (3H5) *respectively,one has*

$$Q_F(a,b) \in \operatorname{cl}(Y) + \operatorname{cl}(Z). \tag{3.5*}$$

Proof. It will suffice applying Proposition 2 to our data. q.e.d.

Some remarks are in order. The obtained statements cannot be in general extended beyond the locally convex setting, as the example in Averbukh and Smolyanov [op.cit.,ch.1,sect.3] shows. On the other hand, the denumerable set A of (3H3) cannot be substituted by a zero Lebesgue measure one; see in this direction McLeod [op.cit.]. Finally, a left counterpart of these developments is possible by the substitution

(3D7) $F(t) = G(-t), \qquad t \in R,$

combined with a reversing of the natural order over the real axis. Since the corresponding versions of these are clear, we do not give details.

4. Some particular aspects

As explicitly stated in Theorem 3 ,a basic regularity condition under which the closure operator is to be dropped in the mean value property (3.4) is

(3H8). This, in particular happens when the locally convex space X fulfils the requirement (2H1). We then have the following practical statement. Let $m \geq 1$ be a fixed natural number, $F : R \to R^m$ a function, and $a, b \in R$ two fixed points with $a < b$. Assume (3H1)+(3H2) are valid, as well as (3H3), for some denumerable part A of $]a, b[$. Take a couple of selections $t \vdash e^{(+)}F(t)$ and $t \vdash D^{(+)}F(t)$ of the multivalued maps $t \vdash \mathcal{E}^{(+)}F(t)$ and $t \vdash \mathcal{D}^{(+)}F(t)$, respectively. Further, let B stand for a zero Lebesgue measure part of $]a, b[$ which includes A.

THEOREM 5. *Under these assumptions, it is the case that*

$$Q_F(a, b) = \sum_{i=1}^{m+1} \lambda_i D^{(+)}F(t_i) + \sum_{j=1}^{m+1} \mu_j e^{(+)}F(s_j), \tag{4.1}$$

where $\{\lambda_i; 1 \leq i \leq m+1\}$, $\{\mu_j; 1 \leq j \leq m+1\}$ *are positive numbers (with, in addition,* $\sum_{i=1}^{m+1} \lambda_i = 1$*) and* $\{t_i; 1 \leq i \leq m+1\}$, $\{s_j; 1 \leq j \leq m+1\}$ *are points in* $]a, b[\setminus B$, *respectively* $[a, b[$.

The number of terms in the convex combination of the right hand side cannot be diminished in general, as simple examples show. But, under the extra condition

(4H1) $\{D^{(+)}F(t);\ t \in]a, b[\setminus B\}$ has at most m connected components,

this is possible (cf. the remarks in Section 2).

An interesting application of these facts may be given along the lines below. Let X be a (real) topological vector space, and $F : R \to X$, a function. Fix a couple of points $a, b \in R$, $a < b$, and assume the general conditions and notations of the preceding section are in use. Let also $p : X \to R$ be a mapping with

(4H2) p is sublinear and continuous.

THEOREM 6. *Under the precised facts,*

$$p(Q_F(a, b)) \leq p(D^{(+)}F(c)) + \mu_1 p(e^{(+)}F(s_1)) + \mu_2 p(e^{(+)}F(s_2)), \tag{4.2}$$

for some c *in* $]a, b[\setminus B$, $\mu_1, \mu_2 \geq 0$ *and* $\{s_1, s_2\}$ *in* $[a, b[$. *Hence, in* *particular, if*

(4H3) $p(e^{(+)}F(t)) \leq 0, \quad t \in [a, b[\,,$

we necessarily have

$$(4.3) \qquad p(Q_F(a,b)) \leq p(D^{(+)}F(c)), \quad \textit{for some } c \textit{ in }]a,b[\, \setminus B.$$

Proof. By the Hahn–Banach theorem, there must be some continuous linear functional x^* with

$$(4.4) \qquad x^*(Q_F(a,b)) = p(Q_F(a,b)), \quad x^*(x) \leq p(x), \; x \in X.$$

Define a function f from R to itself by

$$f(t) = x^*(F(t)), \qquad t \in R.$$

Clearly, Theorem 4 is applicable to f, because

$$\left| \begin{array}{ll} x^*(e^{(+)}F(t)) \in \mathcal{E}^{(+)}f(t), & t \in [a,b[\\ x^*(D^{(+)}F(t)) \in \mathcal{D}^{(+)}f(t), & t \in]a,b[\, \setminus A. \end{array} \right.$$

So, by the conclusion of that statement, one gets via (4.4) (the first part)

$$\left| \begin{array}{lll} p(Q_F(a,b)) & = & \lambda_1 x^*(D^{(+)}F(t_1) + \lambda_2 x^*(D^{(+)}F(t_2)) + \\ & & \mu_1 x^*(e^{(+)}F(s_1)) + \mu_2 x^*(e^{(+)}F(s_2)), \end{array} \right.$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are positive numbers (with $\lambda_1 + \lambda_2 = 1$) and $\{t_1, t_2\}$, $\{s_1, s_2\}$ are points in $]a,b[\, \setminus B$, respectively, $[a,b[$. It will suffice now taking into account (4.4) (the second part) to end the argument. q.e.d.

In particular, the extra condition (4H3) holds under (1H1). So, Theorem 6 includes the one in McLeod [8]; see also Aziz and Diaz [3]. The number of terms in the conical combination of the right hand member cannot be diminished in general, as results from the axample

$$F(t) = \left\{ \begin{array}{ll} 0, & t \leq -1, \; t > 1 \\ 1, & -1 < t \leq 1. \end{array} \right.$$

and (with p=the identity map of R)

$$a = -2, \quad b = 2, \quad A = B = \{-1, 1\}.$$

We finally note that a left counterpart of this result is immediately available, by the methodology of the preceding section. Some historical aspects of the discussed facts may be found in Flett [7,ch.1,sect.11].

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