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ITERATED SUMS OF POLYNOMIAL DIVISORS†

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In praise of His caregivers from the Book

and

In loving memory of

Jacob Thomas Barron & Mamie Green Beard

1908-1997 1912-1984

ABSTRACT

This study reinitiates the 1941 inquiry of E.F. Canaday, the first Ph.D. student of L. Carlitz, which was restricted to $F_2[x] = GF[2,x]$. Since 1974 one of us has focused on perfect, unitary perfect, bi-unitary perfect, and non-unitary perfect polynomials. Here, $F_q[x]$ is partitioned using maximal set-theoretic unions of directed (n,k)-bracelets $A_0 \to A_1 \to \ldots \to A_{k+n} = A_k$ where the monic polynomials $A_0, \ldots, A_{k+n-1}$ are distinct, $n \geq 1$, $k \geq 0$; with $A_i \to A_{i+1}$ denoting $\sigma(A_i) = A_{i+1}$ and $\sigma(A_i)$ the sum of the distinct monic divisors in $F_q[x]$ of the polynomial A_i. New constructions for perfect polynomials $(A \to A)$, disproof of the 1977 conjecture that x divides each perfect polynomial in $F_p[x]$, the existence of infinitely many pairs $A \neq B$ of amicable polynomials $(A \to B \to A)$, and digraph structures of computed maximal unions of non-disjoint bracelets are obtained.

1. INTRODUCTION. E.F. Canaday studied [10] the sum $\sigma(A)$ of the distinct divisors in $F_2[x]$ of polynomials $A = A(x) \in F_2[x]$. Later studies [1]-[9], [15], [17] of various polynomial analogs over F_q, $q = p^d$, $d \geq 1$, of number-theoretic phenomena have yielded numerous results and raised a variety of open questions. Here, Canaday's broad inquiry is reinitiated for $p > 2$. We use his notation and Harary's language [13].

As usual, F_p is represented by the ring of integers *modulo p*. Unless noted otherwise, our discussion is restricted to monic polynomials $A = A(x)$ in the integral domain $F_q[x]$, for which $\sigma(A)$ denotes the sum of the distinct monic polynomials $B \in F_q[x]$ which are divisors of A. We write $A \to C$ whenever $\sigma(A) = C$. Easily, the arithmetic function mapping $F_q[x]$ into the field $F_q(x)$ defined by $\sigma(cA) = c\sigma(A)$, $c \in F_q^*$, is multiplicative; i.e., $\sigma(AB) = \sigma(A)\sigma(B)$ whenever $(A,B) = 1$. Thus σ is completely determined by its values on the non-negative integral powers P^α of the *prime* (monic irreducible) polynomials: $\sigma(P^\alpha) = P^\alpha + P^{\alpha-1} + \cdots + 1 = \frac{P^{\alpha+1}-1}{P-1}$. Since A is the unique monic divisor of A having maximum degree, then σ is both monicicity and degree preserving on the $q^{\deg(A)}$ such monic polynomials A. Taking $\sigma^0(A) = A$, the sequence $\sigma|_A = \{\sigma^i(A)\}_{i\geq 0}$ is eventually perodic. In §2 we study a natural equivalence relation induced on $F_q[x]$ by these *orbits* $\sigma|_A$. Those basic examples and elementary results establish (§3) the infinitude of amicable polynomials $A \neq B$, $A \to B \to A$ over $F_q[x]$, $q = p^d$, for infinitely many primes p and infinitely many integers $d \geq 1$. In §4 we exhibit perfect polynomials over F_{11}, F_{17} which are not divisible by x, refuting a conjecture [2] of the first author.

2. RING-EQUIVALENCE ON $F_q[x]$. Clearly, $F_q[x]$ is partitioned by the scalar multiples over F_q of the monic polynomials $F_q[x]_M$, and the latter are partitioned according to their degrees e, say $F_q[x]_M = \cup_{e\geq 0} F_{q,e}[x]$. Refine these partitions further:

Definition 1. The sequence of monic polynomials $\mathcal{B}(n,k,A)$ given by $A = A_0 \to A_1 \to \ldots \to A_{k+n} = A_k$, where $A_0, A_1, \ldots, A_{k+n-1} \in F_q[x]$ are distinct, is called the *cyclic (n,k)-bracelet generated by A* over F_q. The subsequence $\mathcal{R}(A) = \mathcal{R}(n,A) = \{A_k, \ldots, A_{k+n-1}\}$, is called the *$n$-ring determined by A* if $k > 0$; otherwise, the *n-ring generated* by A. A *bracelet* in $F_q[x]$ is an arbitrary nonempty set-theoretic union of non-disjoint cyclic bracelets generated by monic polynomials $A \in F_q[x]$.■

Definition 2. Polynomials $A, B \in F_q[x]$ are called *ring-equivalent* provided $\mathcal{R}(A) = \mathcal{R}(B)$ set-theoretically, in which case we write $A\mathcal{R}B$.■

Evidently, $\mathcal{R}$-equivalence is an equivalence relation on $F_q[x]_M$ and on each $F_{q,e}[x], e \geq 1$. The $\mathcal{R}$-equivalence classes are the maximal bracelets given by the set-theoretic unions of the non-disjoint iterates of σ "acting" on $F_{q,e}[x]$. I.e., $F_{q,e}[x]$ is partioned by the maximal unions $\overline{\mathcal{B}}$ of non-disjoint orbits $\sigma|_A$, $A \in F_{q,e}[x]$, determined by the finite cyclic semigroup $< \sigma >$. Graph-theoretically: The (obvious) *σ-digraph* $F_q^\sigma[x]$ has infinitely many components, each of which is finite, simple (no multiple edges), and unicyclic with loops, the deletion of whose cycle-edges results in a forest of in-trees having the cycle-vertices as sinks. The structure of finite cyclic semigroups (such as $< \sigma >$) has been understood since Frobenius [11; p. 20], but it seems the orbits under their "action" on sets have been studied only probabilisticly [14], and even Carlitz' classroom lecture notes "The Arithmetic of Polynomials" do not contain applicable arithmetic-function theory. We proceed accordingly.

Theorem 1. *If $A \in F_{q,1}[x], q = p^d, d \geq 1$, then the cyclic bracelet $\mathcal{B}(A)$ is a p-ring.*

Proof. Since A is prime, $\mathcal{B}(A) = A + F_p$ (the additive left coset of F_p in $F_q[x]$).■

An earlier argument [4; Theorem 7] establishes:

Theorem 2. *Let $A(x), B(x) \in F_q[x]$ be monic polynomials, $q = p^d, d \geq 1$. Then $\sigma(A(x)) = \sigma(B(x))$ if and only if $\sigma(A(x+c)) = \sigma(B(x+c))$ for all $c \in F_q$.*■

Note: Distinct translates of polynomials are not necessarily distinct: consider $P(x) = x^3 + 2x + 1 \in F_3[x]$.

If $\sigma(A)$ is a prime then A is a prime-power; this is "best possible": $\sigma((x+i)^2) \in F_5(x)$ is prime, $0 \leq i \leq 4$. Examination of $F_{3,3}[x]$ establishes: If $\sigma(A)$ is a prime-power then "all possibilities" for A are realized. One may apply Theorem 2 to the following "root-results" and translate them across $F_p[x]$ or lift them to $F_q[x]$.

Theorem 3. *Let $x \in F_p[x]$ and $\alpha \geq 1$. Then $x|\sigma[(x+1)^\alpha]$ if and only if $\alpha \equiv -1 (mod\ p)$.*

Proof. Clearly, $\sigma[(x+1)^\alpha] = \sum_{i=0}^{\alpha}(x+1)^i$ has value 0 at $x = 0$ if and only if $\alpha + 1 \equiv 0 (mod\ p)$.■

Corollary 4. *Let $P \in F_p[x]$ be prime, $P \not\equiv 1$ but $P(j) = 1$ for some $j \in F_p$. Then $(x-j)|\sigma(P^\alpha)$ if and only if $\alpha \equiv -1 (mod\ p)$.*■

Theorem 5. *Let $P \in F_p[x]$ be prime with $1 < P(j) < p$ for some $j \in F_p$. Then $(x-j)|\sigma(P^\alpha)$ whenever $\alpha \equiv -1(mod\ (p-1))$.*

Proof. Let $a, j \in F_p$ satisfy $1 < P(j) = a < p$, and consider

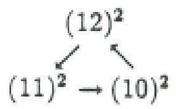
$$(*) \qquad (\sigma[P^\alpha(x)])|_{x=j} = \textstyle\sum_{i=0}^{\alpha} a^i.$$

Since $a \in F_p^*$ has multiplicative order dividing $p-1$, and $\sum_{i=0}^{p-2} a^i \equiv 0(mod\ p)$ from Euler's Theorem, then the number $\alpha + 1 = m(p-1)$ of terms in $(*)$ is sufficient for the claim.■

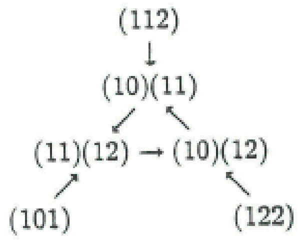
In the course of exhibiting the σ-digraphs which complete this section, we have benefited from discussions with Ann D. Dorris. (Using his personalized/programmable GALOIS II) Rex Matthews has *amicably* provided us with $\sigma(A)$ for $A \in F_{5,4}[x]$ and calculations to double-check our examples. We especially appreciate the dedication of Vickie Mayberry, who has produced our digraphs (using Windows' Paintbrush and TEX). The 600-vertex $F_{5,4}^{\sigma}[x]$ remains under construction (using Maple V). For conciseness represent $A(x) = x^n + a_1x^{n-1} + \cdots + a_n$ by $(1a_1 \cdots a_n)$, and its digraph vertex by the base 10 integer k determined by the base q integer $a_1a_2\ldots a_n$. I.e., $k = A(q) - q^n$ for the *rational integral* polynomial $A(x)$. The distribution of the prime polynomials is exhibited appropriately.

Example 1. $F_{3,2}[x] = \overline{B}_1 \cup \overline{B}_2$ as follows:

$\overline{B}_1$:

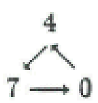


$\overline{B}_2$:

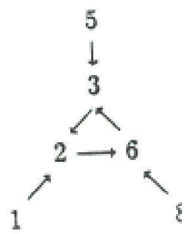


$F_{3,2}^{\sigma}[x]$ is given by:

$\overline{B}_1$:



$\overline{B}_2$:



Example 2. $F_{3,3}[x] = \overline{B}_1 \cup \overline{B}_2 \cup \overline{B}_3 \cup \overline{B}_4$ where

$\overline{B}_1$:

$$\left.\begin{array}{ccc} (1201) & \longrightarrow & (11)(112) \\ (1211) & \longrightarrow & (12)(101) \\ (1222) & \longrightarrow & (10)(122) \\ (1021) & \longrightarrow & (1022) \end{array}\right\} \rightarrow (10)(11)(12) \circlearrowleft$$

$\overline{B}_2$:

$$\begin{array}{ccccc} & (1112) & & (10)^2(12) & \\ & \searrow & \swarrow & \nwarrow & \\ & & (10)(112) & & (12)(122) \\ & \nearrow & \searrow & \nearrow & \\ (11)(122) & & & (11)^3 & \end{array}$$

$\overline{B}_3$:

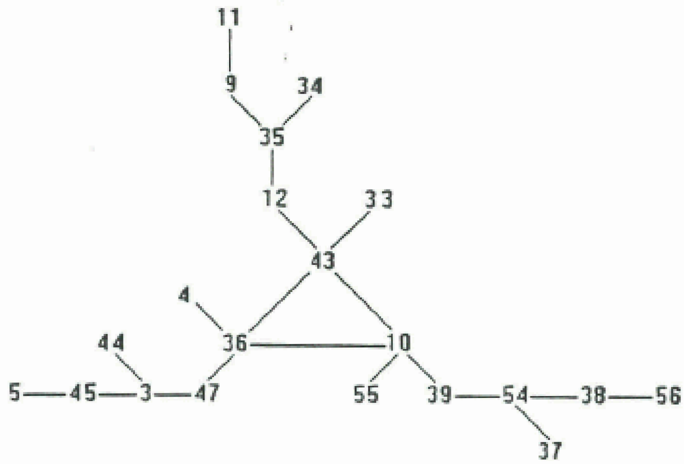
$$\begin{array}{ccccc} & (1102) & & (10)(11)^2 & \\ & \searrow & \swarrow & \nwarrow & \\ & & (10)^2(11) & & (11)^2(12) \\ & \nearrow & \searrow & \nearrow & \\ (12)(112) & & & (12)^3 & \end{array}$$

$\overline{B}_4$:

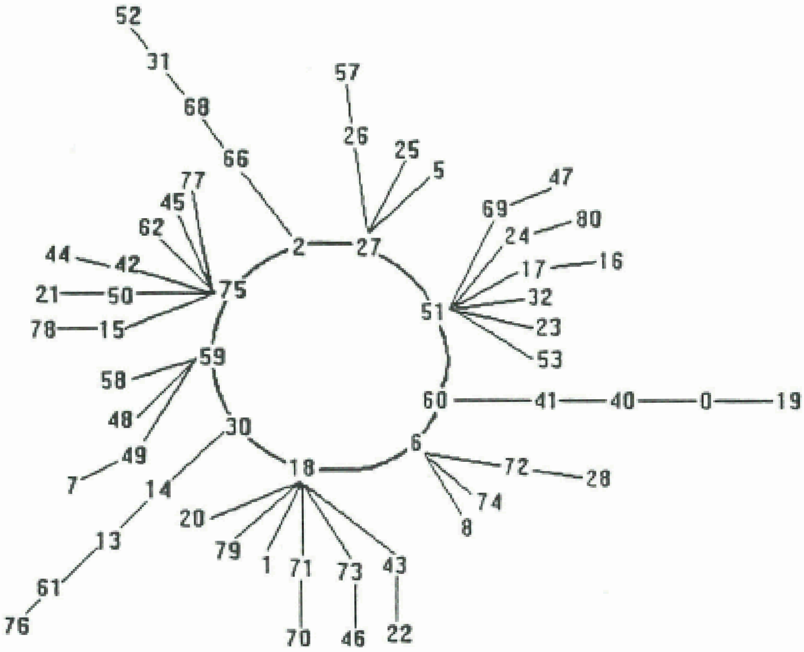
$$\begin{array}{ccccc} & (1121) & & (10)(11)^2 & \\ & \searrow & \swarrow & \nwarrow & \\ & & (10)(11)^2 & & (11)^2(12) \\ & \nearrow & \searrow & \nearrow & \\ (10)(101) & & & (10)^3 & \end{array}$$

Example 3. $F^{\sigma}_{3,4}[x] = \overline{B}_1 \cup \overline{B}_2$ where

$\overline{B}_1$ 21 polynomials; 6/18 primes. :



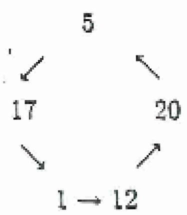
$\overline{B}_2$ 60 polynomials; 12/18 primes. :



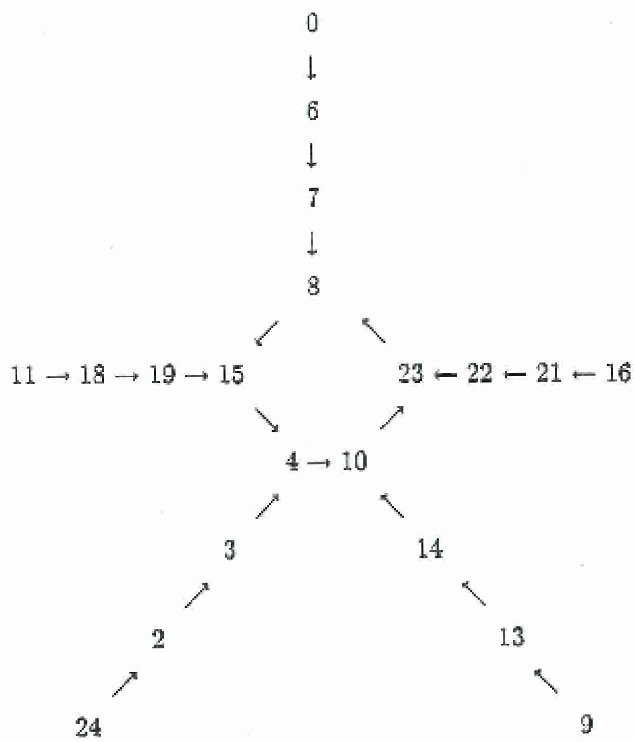
■

Example 4. $F^{\sigma}_{5,2}[x] = \overline{\mathcal{B}}_1 \cup \overline{\mathcal{B}}_2$ where

$\overline{\mathcal{B}}_1$ | 5 polynomials; 0/10 primes. | :

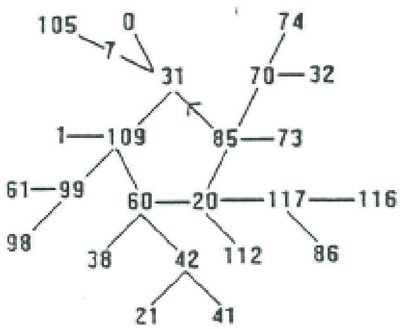


$\overline{\mathcal{B}}_2$ | 20 polynomials; 10/10 primes. | :

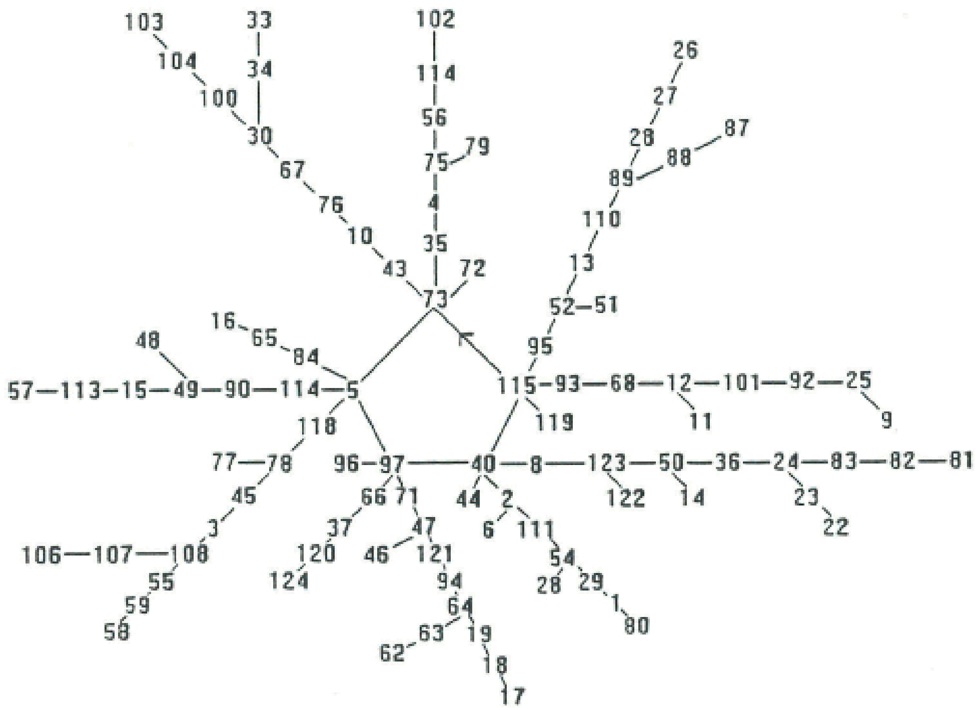


Example 5. $F^{\sigma}_{5,3}[x] = \overline{B}_1 \cup \overline{B}_2$ where

$\overline{B}_1$ [24 polynomials; 4/40 primes.] :



$\overline{B}_2$ [101 polynomials; 36/40 primes.] :



■

3. CYCLIC (p,k)-BRACELETS AND p-RINGS. We now determine the cyclic bracelets generated by specific polynomials of "low" degrees, but have not improved the trivial bounds on cyclic (n,k)-bracelets generated by polynomials of degree $e > 2$ over F_q: $n \geq 1$, $k \geq 0$, $n + k \leq q^e$. First, a general construction technique.

Theorem 6. *The polynomial $A(x) \in F_q[x]$ generates an (n,k)-bracelet if and only if $B_c(x) = A(x+c)$ generates an (n,k)-bracelet for each $c \in F_q$.*

Proof. By finite induction using Theorem 2.■

Our next results extend Theorem 1.

Theorem 7. *Let $c \in F_q$, $q = p^d, d \geq 1$, and $\emptyset \subset \Lambda \subset F_q$. Then $A = \Pi_{i\in\Lambda}(x + c - i)$ generates a p-ring.*

Proof. From Theorem 6, set $c = 0$ w.l.o.g. By Theorem 1, each polynomial $(x - i)$ generates a p-ring. Since $(x - i, x - j) = 1$ for $i \neq j$, and σ is multiplicative, then $A \to \Pi_{i\in\Lambda}\sigma(x-i) = \Pi_{i\in\Lambda}(x-i+1)$. By finite induction, $\sigma^p[\Pi_{i\in\Lambda}(x-i)] = \Pi_{i\in\Lambda}(x-i) = A$. Since $\Lambda \neq F_q$, p is the smallest positive l satisfying $\sigma^l(A) = A$.■

Corollary 8. *Let $A = \Pi_{\alpha\in\Lambda}(x - \alpha) \in F_q[x], q = p^d, d \geq 1$, and let $\Lambda \subset F_q$ have a non-empty proper intersection with at least one of the additive cosets in F_q/F_p. Then A generates a p-ring.*■

Theorem 9. *If $c \in F_q$ and $n \geq 0$, then $(x+c)^{p^n-1}$ generates a p-ring.*

Proof. Set $c = 0$ and note that $x^{p^n-1} \to (x-1)^{p^n-1}$, so $\sigma^p(x^{p^n-1}) = x^{p^n-1}$ and p is the smallest such power of σ.■

Corollary 10. *Let $A = \Pi_{\alpha\in\Lambda}(x-\alpha)^{p^{n(\alpha)}-1} \in F_q[x]$, $q = p^d, d \geq 1$, $n(\alpha) > 0$. If $\Lambda \subset F_q$ has a non-empty proper intersection with at least one additive coset in F_q/F_p, then A generates a p-ring over F_q.*■

Corollary 11. *Whenever $c \in F_q$, $q = p^d, d \geq 1$ and $n(0), \ldots, n(p-1) > 0$ are not all equal, then $A = \Pi_{i=0}^{p-1}(x+c-i)^{p^{n(i)}-1}$ generates a p-ring.*

Proof. Note that $(x-i)^{p^{n(i)}-1} \to (x-i-1)^{p^{n(i)}-1} = [x-(i+1)]^{p^{n(i)}-1}$.■

Hereafter, we restrict our attention to the case $q = p > 2$, and appeal to Legendre symbols $(\frac{a}{p})$ and the distribution of quadratic residues.

Theorem 12. *If $p \equiv 1, 7 (mod\ 12)$, then $x^2 \in F_p[x]$ generates a $(p,1)$-bracelet.*

Proof. For $p \equiv 1, 7(mod\ 12)$, the discriminant of $\sigma(x^2) = x^2 + x + 1$ is $d = -3$. Since $\left(\frac{-3}{p}\right) = 1$ then $x^2+x+1 = (x+j)(x+i)$ for distinct $i, j \in F_p$. By Theorem 7, $(x+j)(x+i)$ generates a p-ring as $p > 3$. Thus x^2 generates a $(p,1)$-bracelet.■

Theorem 13. *If $p \equiv 11, 23, 29, 53, 65, 71 (mod\ 84)$, then $x^2 \in F_p[x]$ generates a $(p,2)$-bracelet.*

Proof. Since $p \equiv 11, 23, 29, 53, 65, 71(mod\ 84)$, then $p \equiv 5, 11(mod\ 12)$, $\sigma(x^2)$ has discriminant $d = -3$, and $\left(\frac{-3}{p}\right) = -1$. Thus $\sigma(x^2)$ is a prime. The discriminant of $\sigma^2(x^2) = x^2 + x + 2$ is $d = -7$ and $p \equiv 1, 9, 11, 15, 23, 25(mod\ 28)$, thus $\left(\frac{-7}{p}\right) = 1$. Hence $\sigma^2(x^2) = (x+i)(x+j)$ for distinct $i, j \in F_p$. By Theorem 7 $(x+j)(x+i)$ generates a p-ring, hence x^2 generates a $(p,2)$-bracelet.■

Theorem 14. *If $p \equiv 1(mod\ 4)$, then $x^3 \in F_p[x]$ generates a $(p,1)$-bracelet.*■

Theorem 15. *If $p \equiv 3(mod\ 8)$ and $p \geq 11$, then $x^3 \in F_p[x]$ generates a $(p,2)$-bracelet.*

Proof. Note that $\sigma(x^3) = x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ and the discriminant of x^2+1 is $d = -4$. Since $p \equiv 3(mod\ 4)$ then $\left(\frac{-1}{p}\right) = -1$, hence x^2+1 is prime. Thus $\sigma^2(x^3) = (x+2)(x^2+2)$. The discriminant of x^2+2 is $d = -8$, and $\left(\frac{-2}{p}\right) = 1$. Hence $\sigma^2(x^3) = (x+2)(x+\sqrt{-2})(x-\sqrt{-2})$. Since its roots are distinct and $p > 3$, then $\sigma^2(x^3)$ generates a p-ring.■

Theorem 16. *If $p > 2$ and $x^2 \in F_p[x]$ generates an (n,k)-bracelet, then $n = p$ and $k \leq \frac{p+1}{2}$.*

Proof. The n-ring of the bracelet is generated by the quadratic $x^2 + x + c$ which "first factors" for $c \in F_p$ under the "ordering" $1 < 2 < \ldots < p$. If $p > 2$ then $n = p$ by Theorem 7, and $x^2 + x + c$ factors if and only if $\left(\frac{1-4c}{p}\right) \neq -1$. Since there are $(p-1)/2$ nonzero quadratic residues in F_p, then there are at most $(p-1)/2$ "consecutive" primes of the form $x^2 + x + c$. Hence $\sigma^{(p+1)/2}(x^2)$ is in the p-ring determined by x^2.■

4. PERFECT POLYNOMIALS NOT DIVISIBLE BY x. Canaday [10] did not find any perfect polynomials over F_2 not divisible by x, and found it plausible that none exist. The exceptional condition proved by him was generalized [2]:

Fact 1. *Every perfect polynomial over $GF(2)$ is either divisible by $x(x-1)$ or else is a perfect square.*■

Fact 2. *If the polynomial $A = \Pi_{i=1}^{k} P_i^{n(i)}$ is perfect over F_p for $n(i) > 0$ and distinct primes P_i having constant terms c_i, then $x|A$ unless all of the following are satisfied:*

(*i*) $(\Pi_{j=0}^{p-1}(x-j), A) = 1$,

(*ii*) $n(i) \equiv 0(mod\ 2)$ *whenever* $c_i = p-1$,

(*iii*) $n(i) \not\equiv -1(mod\ p)$ *whenever* $c_i = 1$,

(*iv*) $n(i) \not\equiv -2(mod\ p-2)$ *whenever* $1 < c_i < p-1$. ■

The conjecture [2] "If A is perfect over F_p then $x|A$" of Beard is false as shown by Link [15]. Her three counter-examples follow in reverse historical order of their constructions (with $(1, a_1, \cdots, a_n) = x^n + a_1 x^{n-1} + \cdots + a_n$).

Example 6. For B=(1,0,1)(1,3,6)(1,8,5)(1,1,4)(1,4,5)(1,7,5)(1,2,2) (1,6,10)(1,9,2)·(1,5,10)(1,10,4)$\in F_{11}[x]$, $A = B^2$ is perfect over F_{11}.■

She also found a perfect polynomial not divisible by x and not a square, but first:

Definition 3. The monic polynomials $A, B \in F_q[x]$ are k-*amicable* over F_q provided $\sigma^k(A) = B$ and $\sigma^k(B) = A$. Whenever A, B are k-amicable and $A \neq B$, then we call A, B a k-*amicable pair* and write $(A,B)_{k-am}$. For $k = 1$ we merely say *amicable*.■

Example 7. Let $A = B^2C \in F_{17}[x]$ where B=(1,4,1)(1,13,1)(1,6,6)(1,15,15)(1,8,13)·(1,14)(1,2,15)(1,10,5)(1,14,12)(1,12,16)(1,16,10)(1,1,10)(1,3,12)(1,5,16)(1,7,5)(1,9,13)·(1,11,16) and C=(1,8,2,12,3)(1,9,2,5,3)(1,12,15,10,9)(1,13,1,6,3)(1,16,6,12,13)(1,12,7)·(1,4,1,11,3)(1,3,9,8,14)(1,11,10,4)(1,7,7,5,1)(1,15,5,13,9)(1,2,5,4,9)(1,6,7,4)·(1,10,7,12,1)(1,14,9,9,14)(1,1,6,5,14)(1,5,15,7,10). $B^2 \to C \to B^2$ and $(B^2, C) = 1$, so A is perfect.■

Finally, Link's Monster (of degree 476):

Example 8. Let $A = B^2C \in F_{17}[x]$ where B=(1,1,1)(1,3,3)(1,16,1)(1,7,13)(1,9,4)·(1,11,14)(1,13,9)(1,15,6)(1,0,5)(1,2,6)(1,4,9)(1,6,14)(1,8,4)(1,10,13)(1,12,7)(1,14,3)(1,5,7)·(1,15,13)(1,4,16)(1,6,4)(1,12)(1,13,16)(1,2,13)(1,10,3)(1,12,14)(1,8,11)(1,14,10)(1,16,8)·(1,1,8)(1,3,10)(1,5,14)(1,7,3)(1,9,11)(1,11,4)(1,7,11)(1,6,12)(1,13,7)(1,12,5)(1,15,4)·(1,14,1)(1,9,2)(1,8,2)(1,5,5)(1,4,7)(1,11,12)(1,10,11)(1,1,16)(1,2,4)(1,3,1)(1,16,16)(1,0,3)·(1,10,1)(1,4,14)(1,9,9)(1,3,8)(1,16,6)(1,15,11)(1,12,12)(1,1,6)(1,11,2)(1,10)(1,6,2)(1,5,12)·(1,8,9)(1,2,11)(1,7,1)(1,13,14)(1,14,8)(1,4,15)(1,16,7)(1,10,2)(1,15,12)(1,3,9)(1,14,9)·(1,9,10)(1,5,13)(1,6,3)(1,1,7)(1,7,2)(1,12,13)(1,11)(1,11,3)(1,2,12)(1,13,15)(1,8,10) and C=(1,2,4,3,3)(1,6,16,4,13)(1,15,4,14,3)(1,14,8,2,13)(1,1,5,13,4)(1,5,14,13,7)(1,9,1,9,6)·(1,13,8,9)(1,11,14)(1,4,9,9)(1,8,1,8,6)(1,12,14,4,7)(1,16,5,4,4)(1,3,8,15,13)(1,7,6,10,6)·(1,11,16,13,13)(1,10,6,7,6)(1,8,13,5,13)(1,15,16,2,6)(1,3,3,16,7)(1,13,12,1,4)(1,6,11,6,6)·(1,11,11,11,6)(1,1,2,9)(1,10,1,16,13)(1,12,9,8,13)(1,2,16,15,6)(1,14,3,1,7)(1,7,1,1,13)·(1,6,14)(1,5,9,9,13)(1,4,12,16,4)(1,9,13,12,3)(1,16,15,9). Then $\Pi_{i=0}^{16}((x+i)^2+5)^2 \to \Pi_{i=0}^{16}((x+i)^4+11(x+i)^2+14) \to \Pi_{i=0}^{16}((x+i)^2+12)^2 \to \Pi_{i=0}^{16}((x+i)^2+3)^2 \to \Pi_{i=0}^{16}((x+i)^2+10)^2 \to \Pi_{i=0}^{16}((x+i)^2+11)^2 \to \Pi_{i=0}^{16}((x+i)^4+6(x+i)^2+14) \to \Pi_{i=0}^{16}((x+i)^2+5)^2$. ■

Example 7 shows a new technique for constructing perfect polynomials:

Theorem 17. *Let $A, B \in F_q[x]$ and $(A,B) = 1$. If $(A,B)_{am}$ then AB is perfect.* ■

Corollary. *If $A \in F_q[x]$ generates an n-ring of pairwise relatively prime A_i, then $\Pi_{i=0}^{n-1} A_i$ is perfect.* ■

The converse of Theorem 17 is false:

Example 9. Let $A = B^2C(x^{17}-x) \in F_{17}[x]$ with B, C from Example 8. Since B^2C and $x^{17} - x = \Pi_{i=0}^{16}(x-i)$ are perfect, and $(B^2C, x^{17}-x) = 1$, then A is perfect. No factorization $A = DE$ with $(D,E) = 1$ yields $(D,E)_{am}$. ■

5. Amicable Pairs and Other even-Rings. The amicable pairs $(A, \sigma(A))_{am}$ over F_p given in Table I for $3 \le p \le 17$ were obtained by hand using [9] or on a PC using the GALOIS package of Lidl, Matthews, and Wells. For brevity, the five amicable pairs and infinite class over $GF(2)$ which Canady discovered [10] are not included, and those constructible from our theoretical results are omitted. For conciseness, let $x_i = x + i$.

TABLE 1

p	$(A,B)_{am}$
3	$A = (x^3-x)^3, B = (x^3-x)\Pi_{i=0}^{2}(x_i^2+1)$
3	$A = (x^3-x)^4\Pi_{i=0}^{2}(x_i^3+2x_i^2+1), B = (x^3-x)\Pi_{i=0}^{2}(x_i^2+1)\Pi_{i=0}^{2}\sum_{j=0}^{4} x_i^j$
3	$A = (x^3-x)^7\Pi_{i=0}^{2}(x_i^4+2x_i^2+2), B = (x^3-x)^5\Pi_{i=0}^{2}(x_i^2+1)^3$ $\cdot\Pi_{i=0}^{2}(x_i^4+2x_i^3+x_i^2+2x_i+1)$
3	$A = (x^3-x)^9\Pi_{i=0}^{2}(x_i^3+2x_i^2+1)\Pi_{i=0}^{2}(x_i^3+x_i^2+2), B = (x^3-x)^5$ $\cdot\Pi_{i=0}^{2}(x_i^2+1)\Pi_{i=0}^{2}\sum_{j=0}^{4} x_i^j\Pi_{i=0}^{2}(x_i^4+2x_i^3+x_i^2+2x_i+1)$

p	$(A,B)_{am}$
3	$A=(x^3-x)^{10}\Pi_{i=0}^{2}(x_i^2+1)(x_i^3+x_i^2+2), B=(x^3-x)^5$ $\cdot\Pi_{i=0}^{2}(x_i^5+x_i^4+2x_i+1)\Pi_{i=0}^{2}(x_i^5+2x_i^4+2x_i^3+2x_i+1)$
3	$A=(x^3-x)^{19}\Pi_{i=0}^{2}(x_i^3+x_i^2+2)(x_i^3+2x_i^2+1), B=(x^3-x)^5$ $\cdot\Pi_{i=0}^{2}(x_i^2+1)^2(x_i^4+x_i^2+2x_i+1)\Pi_{i=0}^{2}(x_i^4+x_i^2+x_i+1)(x_i^4+x_i+2)$ $\cdot\Pi_{i=0}^{2}(x_i^4+2x_i+2)$
3	$A=(x^3-x)^{27}\Pi_{i=0}^{2}(x_i^2+1)^2\Pi_{i=0}^{2}\sum_{j=1}^{6}(x_i^j+2)\Pi_{i=o}^{2}(x_i^6+2x_i^5+x_i^4+2x_i^3+2x_i+1)$ $B=(x^3-x)^{17}\Pi_{i=0}^{2}(x_i^2+1)\Pi_{i=0}^{2}\sum_{j=0}^{6}x_i^j\Pi_{i=0}^{2}(x_i^6+x_i^5+x_i^3+1)$ $\cdot\Pi_{i=0}^{2}(x_i^6+2x_i^5+2x_i^3+2x_i+1)\Pi_{i=0}^{2}(x_i^6+2x_i^5+x_i^4+2x_i^3+x_i^2+2x_i+1)$
5	$A=(x^5-x)^5\Pi_{i=0}^{4}(x_i^4+1), B=(x^5-x)\Pi_{i=0}^{4}(x_i^2+2)^2(x_i+3)$
5	$A=(x^5-x)^{13}\Pi_{i=0}^{4}(x_i^2+2)^2(x_i^4+3)\Pi_{i=0}^{4}\sum_{j=1}^{6}(x_i^j+2)\Pi_{i=0}^{4}(x_i^6+3x_i^5+x_i^4+4x_i+3),$ $B=(x^5-x)^9\Pi_{i=0}^{4}(x_i^4+2)(x_i^4+x_i+4)\Pi_{i=0}^{4}(x_i^4+4x_1+4)\Pi_{i=0}^{4}\sum_{j=0}^{6}(x_i^j)$ $\cdot\Pi_{i=0}^{4}(x_i^6+3x_i^5+x_i^4+4x_i+2)$
5	$A=(x^5-x)^{15}\Pi_{i=0}^{4}(x_i^2+3)^2(x_i^4+3), B=(x^5-x)^9\Pi_{i=0}^{4}(x_i^2+2)(x_i^2+3)(x_i^4+2)(x_i^4+3)$
7	$A=(x^7-x)^2\Pi_{i=0}^{6}\sum_{j=1}^{4}(x_i^j+2), B=(x^7-x)^2\Pi_{i=0}^{6}(x_i^2+4)\Pi_{i=0}^{6}\sum_{j=0}^{4}x_i^j$
11	$A=(x^{11}-x)^6, B=\Pi_{i=0}^{10}(x_i^3+7x_i^2+6x_i+10)\Pi_{i=0}^{10}(x_i^3+5x_i^2+4x_i+10)$
13	$A=(x^{13}-x)^4\Pi_{i=0}^{12}(x_i^2+11)\Pi_{i=0}^{12}\sum_{j=1}^{4}x_i^j+2), B=(x^{13}-x)^3$ $\cdot\Pi_{i=0}^{12}(x_i^3+7x_i+7x_i+7)\Pi_{i=0}^{12}\sum_{j=0}^{4}x_i^j$
13	$A=(x^{13}-x)^6\Pi_{i=0}^{12}(x_i^2+6)(x_i^2+8), B=(x^{13}-x)^2$ $\cdot\Pi_{i=0}^{12}(x_i^2+2)(x_i^2+5)(x_i^2+7)(x_i^2+11)$
13	$A=(x^{13}-x)^7\Pi_{i=0}^{12}(x_i^2+6)(x_i^2+8), B=(x^{13}-x)^5\Pi_{i=0}^{12}(x_i^2+5)(x_i^2+7)(x_i^2+8)$
13	$A=(x^{13}-x)^8\Pi_{i=0}^{12}(x_i^3+11), B=(x^{13}-x)^5\Pi_{i=0}^{12}(x_i^3+4)(x_i^3+10)$
17	$A=\Pi_{i=0}^{16}(x_i^2+14)^2, B=\Pi_{i=0}^{16}(x_i^4+12x_i^2+7)$ ■

To construct amicable pairs, recall Example 9, Theorem 17, and from [2],[8]:

Fact 3. *The polynomial* $A=\Pi_{i=0}^{p-1}(x-i)^{n(i)}\in F_p[x]$ *is perfect if and only if* $n(0)=\ldots=n(p-1)=Np^n-1$ *where* $N|(p-1)$ *and* $n\geq 0$.■

Theorem 18. *Let* $A,B,C\in F_q[x]$ *be pairwise relatively prime. Then* A *is perfect and* $(B,C)_{am}$ *if and only if* $(AB,AC)_{am}$.

Proof. The necessity of $(AB,AC)_{am}$ is evident. Conversely, we have $\sigma(A)\sigma(B)=AC$ and $\sigma(A)\sigma(C)=AB$, from which $\sigma(A)\sigma(B)\sigma(C)=AC\sigma(C)=AB\sigma(B)$. So $C\sigma(C)=B\sigma(B)$, $C|\sigma(B)$, and $B|\sigma(C)$. Since $(AB,AC)_{am}$ and σ is degree-preserving, then $\deg B=\deg C$. Since $\sigma(B)$ is the unique monic divisor of itself having maximum degree then $C=\sigma(B)$. By symmetry, $B=\sigma(C)$. Since $B\neq C$ then $(B,C)_{am}$. Finally, $\sigma(A)\sigma(B)=AC=A\sigma(B)$ so $\sigma(A)=A$.■

Example 10. There exist infinitely many amicable pairs of polynomials over F_q for $q=17^d$, d odd, by Fact 3 and Theorem 18: B^2,C of Example 8 are relatively prime to $x^{17}-x$; $n\geq 0$ is arbitrary in Fact 3; and the prime factors of B^2,C remain prime over F_q [12].■

From Dirichlet's theorem on the infinitude of primes in an arithmetic progression, the next three results each establish that $(A,\sigma(A))_{am}$ exist for infinitely many odd primes p.

Lemma 1. *If $p \geq 3$ and $c, i, j \in F_p$ with $i \neq j$, the translates $(x+i)^2+(x+i)+c$ and $(x+j)^2+(x+j)+c$ are distinct.*■

Theorem 19. *The polynomials $A=(x^p-x)^2$, $\sigma(A) \in F_p[x]$ are amicable if and only if $p \equiv 11, 23, 29, 53, 65, 71 (mod\ 84)$.*

Proof. If $p=2$, then $A=(x^2-x)^2 \rightarrow x^4+x+1 \rightarrow x(x+1)^3$ so that A and $\sigma(A)$ are not amicable. If $p=3$, then $(x^3-x)^2 \leftrightarrow (x^3-x)^2$ and $A=(x^3-x)^2$ is perfect. Thus let $p \geq 5$ hereafter. Note that the discriminant of $\sigma(x^2)=x^2+x+1$ is $d=-3$ and $(\frac{3}{p}) \neq 0$. If $(\frac{-3}{p})=1$, then $\sigma(x^2)=(x+i)(x+j)$ for distinct $i, j \in F_p$. Thus from Lemma 1, $\sigma((x^p-x)^2)=(x^p-x)^2$ so A is perfect, hence A and $\sigma(A)$ are not an amicable pair over F_p. Hence assume $(\frac{-3}{p})=-1$, which occurs if and only if $p \equiv 5, 11 (mod\ 12)$. Then x^2+x+1 is prime over F_p. Thus $(x^p-x)^2 \rightarrow \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+1)$ by Theorem 2 and is in canonical form by Lemma 1. Assume A and $\sigma(A)$ are amicable over F_p. Then $\sigma^2(x^2)|(x^p-x)^2$, so $\sigma^2(x^2)=\sigma(x^2+x+1)=x^2+x+2$ factors. Since the discriminant of x^2+x+2 is $d=-7$, then x^2+x+2 factors if and only if $(\frac{-7}{p})=1$ or $p=7$. Since $p \equiv 5, 11 (mod\ 12)$, then $p \neq 7$. Hence $(\frac{-7}{p})=1$, which occurs if and only if $p \equiv 1, 3, 9, 19, 25, 27 (mod\ 28)$. Solving the system of congruences

$$\begin{cases} p \equiv 5, 11 (mod\ 12) \\ p \equiv 1, 3, 9, 19, 25, 27 (mod\ 28) \end{cases} \quad \text{yields } p \equiv 11, 23, 29, 53, 65, 71 (mod\ 84).$$

Conversely, assume $p \equiv 11, 23, 29, 53, 65, 71 (mod\ 84)$. Then $p \equiv 5, 11 (mod\ 12), (\frac{-3}{p})=-1$, and x^2+x+1 is prime, so that $(x^p-x)^2 \rightarrow \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+1)$. By Theorem 2 and Lemma 1, $\sigma(\Pi_{i=0}^{p-1}((x+i)^2+(x+i)+1))=\Pi_{i=0}^{p-1}(\sigma((x+i)^2+(x+i)+1))$. Since the discriminant of x^2+x+1 is $d=-7$ and $p \equiv 1, 3, 9, 19, 25, 27 (mod\ 28)$, then $(\frac{-7}{p})=1$, so that $\sigma(x^2+x+1)=(x+j)(x+l)$ for distinct $l, j \in F_p$. By Theorem 2 and Lemma 1, $\sigma[\Pi_{i=0}^{p-1}(x+i)^2+(x+i)+1]=\Pi_{i=0}^{p-1}[\sigma((x+i)^2+(x+i)+1)]=\Pi_{i=0}^{p-1}(x+j+i)(x+l+i)=(x^p-x)^2$ and we are done.■

Lemma 2. *For $p \geq 3$, the translates of $x^2+1 \in F_p[x]$ are distinct.*■

Theorem 20. *The polynomials $A=(x^p-x)^3$, $\sigma(A) \in F_p[x]$ form an amicable pair over F_p if and only if $p \equiv 3 (mod\ 8)$.*■

Theorem 21. *For $A=(x^p-x)^2$, the polynomials $B=A\sigma^2(A)$ and $\sigma(B)$ are an amicable pair over F_p if and only if $p \equiv$17,83,167,173,227,293,437,503,563,593,677,857,887, 923,1007,1097,1217,1223,1403,1427,1487,1517,1553,1613,1823,1847,2063,2147,2243,2273, 2327,2393,2477,2483,2537,2603,2747,2813,2873,2903,2987,3167,3197,3233,3317,3407,3527, 3533,3713,3737,3797,3827,3863,3923,4133,4157,4373,4457,4553,4583 (mod 4620).*

Proof. The result will follow from Theorem 22 and Theorem 23, as we now argue. The displayed polynomial A generates a 4-ring over F_p if and only if p satisfies the above congruence. In the proof of Theorem 23, the elements of the 4-ring generated by A are shown to be pairwise relatively prime. Hence by Theorem 22, the polynomials $B=A\sigma^2(A)$ and $\sigma(B)$ are amicable.■

Theorem 22. *Let $A_0 \in F_q[x]$ generate the even n-ring $A_0 \to A_1 \to \ldots \to A_{n-1} \to A_n = A_0$. If A_i, A_{i+2} relatively prime for $0 \le i < n$ and $B = \Pi_{k=1}^{\frac{n}{2}} A_{2k-1}$ and $C = \Pi_{k=0}^{\frac{n}{2}-1} A_{2k}$, then $(B, C)_{am}$.* ■

Example 10. For $A_0 = (x^{17} - x)^2 \in F_{17}[x]$, for $x_i = x + i$: $A_0 = (x^{17} - x)^2 \to \Pi_{i=0}^{16}(x_i^2 + x_i + 1) = A_1 \to \Pi_{i=0}^{16}(x_i^2 + x_i + 2) = A_2 \to \Pi_{i=0}^{16}(x_i^2 + x_i + 3) = A_3 \to \Pi_{i=0}^{16}(x_i^2 + x_i + 4) = \Pi_{i=0}^{16}[(x + 6 + i)(x + 12 + i)] = A_0$. ■

Theorem 23. The polynomial $A = (x^p - x)^2 \in F_p[x]$ generates a 4-ring over F_p if and only if $p \equiv$17,83,167,173,227,293,437,503,563,593,677,857,887,923,1007,1097,1217,1223, 1403,1427,1487,1517,1553,1613,1823,1847,2063,2147,2243,2273,2327,2393,2477,2483,2537, 2603,2747,2813,2873,2903,2987,3167,3197,3233,3317,3407,3527,3533,3713,3737,3797,3827, 3863,3923,4133,4157,4373,4457,4553,4583 (*mod* 4620).

Proof. If $p = 2$, then $(x^2 - x)^2 \to x^4 + x + 1 \to x^3(x + 1) \leftrightarrow x(x + 1)^3$. Thus $(x^2 - x)^2$ generates a $(1, 2)$-bracelet rather than a 4-ring. Hence assume $p \ge 3$. The discriminant of $\sigma(x^2) = x^2 + x + 1$ is $d = -3$. If $p = 3$, then $(x^3 - x)^2 \leftrightarrow (x^3 - x)^2$, so A is perfect over F_3. Hence assume $p \ge 5$. If $(\frac{-3}{p}) = 1$, then $x^2 + x + 1$ has distinct roots over F_p and factors as $x^2 + x + 1 = (x + l)(x + j)$. By Theorem 2 and Lemma 2, we have $\sigma((x^p - x)^2) = \Pi_{i=0}^{p-1}((x + i)^2 + (x + i) + 1) = \Pi_{i=0}^{p-1}(x + l + i)(x + j + i) = (x^p - x)^2$, so that A is perfect over F_p. Hence assume $p \ge 5$ and $(\frac{-3}{p}) = -1$, which occurs if and only if $p \equiv 5, 11 (mod\ 12)$. Then $x^2 + x + 1$ is prime over F_p. By Theorem 2 and Lemma 1, we have the canonical factorization $\sigma(x^p - x)^2 = \Pi_{i=0}^{p-1}((x + i)^2 + (x + i) + 1)$. The discriminant of $x^2 + x + 2$ is $d = -7$. Since $p \equiv 5, 11 (mod\ 12)$, then $p \ne 7$ so $p \ge 11$. By the proof of Theorem 19, if $(\frac{-7}{p}) = 1$ then $(A, \sigma(A))_{am}$ so that A does not generate a 4-ring. Hence assume $(\frac{-7}{p}) = -1$, which occurs if and only if $p \equiv 3, 5, 13, 17, 19, 27 (mod\ 28)$. Then $x^2 + x + 2$ is prime over F_p. By Theorem 2 and Lemma 1, $\sigma^2(A)$ has the canonical factorization $\sigma^2(A) = \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+2)$, and $\sigma^3(A) = \Pi_{i=0}^{p-1}(\sigma((x+i)^2+(x+i)+2))$. Note that the discriminant of x^2+x+3 is $d = -11$. If $p = 11$, then $(A, \sigma(A))_{am}$ by Theorem 19, and A does not generate a 4-ring. If $(\frac{-11}{p}) = 1$, then $x^2 + x + 3$ has distinct roots over F_p and factors as $\sigma^3(x^2) = x^2 + x + 3 = (x + l)(x + j)$. Thus by Theorem 2 and Lemma 2, $\sigma^3(A) = \Pi_{i=0}^{p-1}((x + i)^2 + (x + i) + 3) = \Pi_{i=0}^{p-1}(x + l + i)(x + j + i) = (x^p - x)^2$ so that A generates a 3-ring over F_p. Hence assume $(\frac{-11}{p}) = -1$, so that $x^2 + x + 3$ is prime over F_p, which occurs if and only if $p \equiv 7, 13, 17, 19, 21, 29, 35, 39, 41, 43 (mod\ 44)$. By Theorem 2 and Lemma 1, $\sigma^3(A) = \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+3)$ and $\sigma^4(A) = \Pi_{i=0}^{p-1}(\sigma((x+i)^2+(x+i)+3)) = \Pi_{i=0}^{p-1}((x + i)^2 + (x + i) + 4)$, and the discriminant of $x^2 + x + 4$ is $d = -15$. Assume that A generates a 4-ring over F_p. Then $\sigma^4(x^2) | (x^p - x)^2$. Thus $x^2 + x + 4$ factors. Hence $(\frac{-15}{p}) = 1$, which occurs if and only if $p \equiv 5, 7, 11, 13, 29, 37, 43, 59 (mod\ 60)$. Solving

$$\begin{cases} p \equiv 5, 11 (mod\ 12) \\ p \equiv 3, 5, 13, 17, 19, 27 (mod\ 28) \\ p \equiv 7, 13, 17, 19, 21, 29, 35, 39, 41, 42 (mod\ 44) \\ p \equiv 5, 7, 11, 13, 29, 37, 43, 59 (mod\ 60) \end{cases}$$

yields the desired congruence.

Conversely, let p satisfy the congruence. From earlier arguments: $A \to \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+1) \to \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+2) \to \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+3) \to \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+4)$. Since $p \equiv 5, 7, 11, 13, 29, 37, 43, 59 (mod\ 60)$ then $(\frac{-15}{p}) = 1$. Hence $\sigma^4(x^2)$ has distinct roots in F_p, say $\sigma^4(x^2) = (x+l)(x+j)$. By Theorem 2 and Lemma 1, $\sigma^4(A) = \Pi_{i=0}^{p-1}((x+i)^2+(x+i)+4) = \Pi_{i=0}^{p-1}(x+l+i)(x+j+i) = A$. ■

Corollary 24. *There exist 2-amicable pairs of polynomials over F_p for infinitely many primes p.*■

On mimicing Example 12 using amicable pairs from Table I, Theorems 19,20,21 and perfect polynomials displayed in or constructible from [2],[4],[8], there is evidence for this

Conjecture. *For all primes p and integers $k, d \geq 1$, there exist infinitely many $(A, B)_{k-am}$ over F_q, $q = p^d$.*■

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