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SAMPLE CONTROLLABILITY OF GENERAL NONLINEAR STOCHASTIC SYSTEMS

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1. Introduction

Stochastic control theory is a stochastic generalization of classsical control theory. The problem of controllability of nonlinear stochastic systems has been discussed by many authors (e.g., see [2, 8]). For controllability of classical nonlinear system, among other methods, fixed point techniques are widely used as a tool [9]. Anichini [1], Balachandran [3], Dauer [5], and Yamamoto [9], respectively, studied the controllability of classical nonlinear systems by means of Schaefer's theorem, Darbo's theorem, and Fan's theorem (see also [4]).

The purpose of this paper is to consider the controllability of nonlinear stochastic systems. The variation of parameters formula for stochastic differential equations [7] and random fixed point theorem [6] are used to get suitable controllability conditions. Further comparision theorems for controllability of nonlinear stochastic systems are obtained.

2. Preliminaries

Consider the process described by the differential equation

$$x'(t,\omega) = A(x,u,t,\omega) + B(x,u,t,\omega)u(t,\omega) + f(x,u,t,\omega) \tag{1}$$

where $B(x,u,t,\omega)$ is the product measurable random matrix function defined on $R^n \times R^m \times J \times \Omega$ into $R^{n\times m}$. Assume that $A, f \in M\big[R^n \times R^m \times J,\ R[\Omega, R^n]\big]$ and $A(x,u,t,\omega)$, $B(x,u,t,\omega)$ and $f(x,u,t,\omega)$ are almost surely (a. s.) sample continuous where $J = [t_0, t_1]$.

Here the notations and definitions are adopted from reference [7] and it is assumed that all inequalities and relations involving random quantities are valid with probability one.

The following two hypothesis are also assumed:

$$|\ A(x,u,t,\omega)\ | \le k_a(t,\omega),\ \|\ B(x,u,t,\omega)\ \| \le k_b(t,\omega), \tag{2}$$

$$|\ f(x,u,t,\omega)\ \ \le g(\ |\ x\ |, |\ u\ |, t,\omega), \tag{3}$$

where $k_a, k_b \in IB\big[J, R[\Omega, R_+]\big]$, $g \in IB\big[R_+^n \times R_+^m \times J, R[\Omega, R_+]\big]$, $g(\alpha,\beta,t,\omega)$ is sample continuous and nondecreasing for any $\alpha > 0$, $\beta > 0$, $A(x,u,t,\omega)$ be twice continuously (randomly) differentiable in $x(t,\omega)$, and $u(t,\omega)$ be sample continuous random function.

Here $\|\cdot\|$ is a norm on Banach space and $|\cdot|$ is the Euclidean norm. In this paper a set of sufficient conditions is developed to ensure that the system state $x(t,\omega)$ can be transformed from $x(t_0,\omega) = x_0(\omega)$ to $x_1(\omega)$ in the alloted time $t_1 - t_0$.

Definition. The stochastic system (1) is sample controllable from $(x_0(w), t_0)$ to $(x_1(w), t_1)$ if for some random control $u(t,w)$ on J, the sample solution of (1) with $x(t_0,w) = x_0(w)$ is such that $x(t_1,w) = x_1(w)$. If the system (1) is sample controllable for all $x_0(w)$ at $t = t_0$ and for all $x_1(w)$ at $t = t_1$, it is called completely sample controllable on J.

Choose $x_0(\omega), x_1(\omega)$ and consider how to determine an appropriate control $u(t,\omega)$ which steers the solution of the system (1) with $x(t_0,\omega)$ to $x(t_1,\omega) = x_1(\omega)$. Since the system (1) is nonlinear, it is not known whether such a control exists. Therefore, we consider the system (4), instead of the system (1).

$$x'(t,\omega) = A(y,v,t,\omega) + B(y,v,t,\omega)u(t,\omega) + f(y,v,t,\omega) \tag{4}$$

where y, v are continuous functions with appropriate dimensions. Then the system (4) is a linear control system with time functions $A(y,v,t,\omega), B(y,v,t,\omega)$ and $f(y,v,t,\omega)$; a sufficient condition for the system (4) to be completely sample controllable is that

$$det\ G(t_0,t,y,v,\omega) \geq c, \text{ for some positive constant } c, \text{ where} \tag{5}$$

$$G(t_0,t_1,y,v,\omega) = \int_{t_0}^{t_1} H(t,t,y,v,\omega)H^T(t,\tau,y,v,\omega)d\tau$$

$$H(t,\tau,y,v,\omega) = \Phi(t,\tau,y,v,\omega)B(y,v,\tau,\omega)$$

$$\phi(t,s,y,v,w) = \frac{\partial y\big(t,s,x_0(w),w\big)}{\partial x_0}$$

$$\partial\Phi(t,t_0,y,u,\omega)/\partial t = \big[\partial A(y,u,t,\omega)/\partial y\big]\Phi$$

$$\Phi(t,t,y,v,\omega) = I, \text{ the identity matrix.}$$

Here the T denotes the matrix transpose. The solution of the system (4) with $x(t_0,\omega) = x_0(\omega)$ is given by [7]

$$x(t,\omega) = y\big(t,t_0,x_0(\omega)\big) + \int_{t_0}^{t} \Phi(t,\tau,y,v,\omega)B(y,v,\tau,\omega)u(\tau,\omega)d\tau \\ + \int_{t_0}^{t} \Phi(t,\tau,y,v,\omega)f(y,v,\tau,\omega)d\tau \tag{6}$$

where $y\big(t,t_0,x_0(\omega)\big)$ is the unique solution of the equation $y'(t,\omega) = A(y,v,t,\omega)$ with $y\big(s,s,x_0(\omega)\big) = x_0(\omega)$. If the system (4) satisfies the condition (5) then one of the controls which steers the state (6) to a given $x_1(\omega)$ at time t_1 is given by

$$u(t,\omega) = H^T G^{-1}(t_0,t_1,y,v,\omega)\Big[x_1(\omega) - y(t_1,t_0,x_0(\omega)) \\ - \int_{t_0}^{t_1} \Phi(t_1,\tau,y,v,\omega)f(y,v,t,\omega)d\tau. \tag{7}$$

Substituting equation (7) into (6) it follows that $x(t_1,\omega) = x_1(\omega)$. If appropriately selected vectors y, v agree with x, u which result from equations (6) and (7) respectively, then these vectors are also solutions of the original problem for system (1) and the controllability problem for system (1) becomes an existence problem of a fixed point for equations (6) and (7). But if there is at least one set of fixed points for equations (6) and (7), then this solution is also obtained from equations (7) and (8)

$$x(t,\omega) = y\big(t,t_0,x_0(\omega)\big) + \int_{t_0}^{t} \Phi(t,\tau,y,v,\omega)B(y,v,\tau,\phi)v(\tau)d\tau \\ + \int_{t_0}^{t} \Phi(t,\tau,y,v,\omega)f(y,v,\tau,\omega)d\tau \tag{8}$$

Equations (7) and (8) are also considered as nonlinear operator equations which assign $(y,v) \in C[J,R^{n+m}]$ to $(x,u) \in C[J,R^{n+m}]$ where $C[J,R^{n+m}]$ is the collection of continuous functions defined on J with values in R^{n+m}. Thus, the original controllability problem becomes a fixed point problem for the operator F

$$(x(t,\omega),u(t,\omega)) = F(y(t),v(t),\omega).$$

This nonlinear operator F is obviously continuous on $C[J,R^{n+m}]$. Furthermore, if there exists a closed bounded convex subset S of $C[J,R^{n+m}]$ such that the operator F is invariant for S, i.e.,

$$(x(t,\omega),u(t,\omega)) = F(y(t),v(t),\omega) \in S \; for\ any\ y(t),v(t) \in S.$$

Then, as derived from equations (7) and (8), the operator F is bounded and equicontinuous. Hence by Schauder's theorem, there exists at least one fixed point for F. Thus, we have the following theorem.

Theorem 1. *If there exists a closed bounded convex subset S of $C[J,R^{n+m}]$ such that the random operator F satisfies $F(\omega,S) \subset S$, for each $\omega \in \Omega$, then the nonlinear stochastic system (1) which satisfies condition (5) is completely sample controllable.*

3. Comparison Theorems for Sample Controllability

Now we examine the conditions such that there exists a subset S which satisfies Theorem

1. Define $S(\omega)$ to be a set

$$S(\omega) = \Big\{(y, v) \in C\big[J, R^{n+m}\big] : \mid y(t) \mid \leq \alpha(t, \omega), \mid v(t) \mid \leq \beta(t, \omega)\Big\} \tag{9}$$

where $\alpha, \beta \in IB\big[J, R(\Omega, R_+)\big]$. Then it follows that

$$\begin{aligned}
\mid x(t,\omega) \mid \; &\leq \mid y\big(t, t_0, x_0(\omega)\big) \mid + \int_{t_0}^{t} \|\Phi(t, \tau, y, v, \omega)\| \; \|B(y, v, \tau, \omega)\| \; \mid v(\tau) \mid d\tau \\
&\quad + \int_{t_0}^{t} \|\Phi(t, \tau, y, v, \omega)\| f\big(y(\tau), v(\tau), \tau; \omega\big) d\tau \\
&\leq \mid x_0(\omega) \mid + k_a(\omega) + exp\; k(\omega) \int_{t_0}^{t} k_b(\tau, \omega) \mid v(\tau) \mid d\tau \\
&\quad + \; exp\; k(\omega) \int_{t_0}^{t} g\big(\mid y(\tau) \mid, \mid v(\tau) \mid, \tau, \omega\big) d\tau \\
&\leq \mid x_0(\omega) \mid + k_a(\omega) + \; exp\; k(\omega) \int_{t_0}^{t} k_b(\tau, \omega) \mid v(\tau) \mid d\tau \\
&\quad + \; exp\; k(\omega) \int_{t_0}^{t} g\big(\mid y(\tau) \mid, \mid v(\tau) \mid, \tau, \omega\big) d\tau \\
&\leq a_0(\omega) + a_1(\omega) \int_{t_0}^{t} k_b(\tau, \omega) \beta(\tau, \omega) d\tau + a_1(\omega) \int_{t_0}^{t} g\big(\alpha(\tau, \omega), \beta(\tau, \omega), \tau, \omega\big) d\tau
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\mid y(t, t_0, x_0(\omega) \mid &\leq \mid x_0(\omega) \mid + \int_{t_0}^{t_1} \|A(y, v, s, \omega)\| ds \\
&\leq \mid x_0(\omega) \mid + \int_{t_0}^{t_1} k_a(s, \omega) ds \\
&= \mid x_0(\omega) \mid + k_a(\omega),
\end{aligned}$$

$$a_0(\omega) = k_a(\omega) + \mid x_0(\omega) \mid,$$

$$\|\Phi(t, t_0, y, v, \omega)\| \leq \; \|I\| + \int_{t_0}^{t_1} \|\partial A(y, v, s, \omega)/\partial y\| \; \|\Phi(s, t_0, y, v, \omega)\| ds$$

$$\leq exp \int_{t_0}^{t_1} K(s,w)ds.$$

$$= \ exp\ k(\omega), \quad \text{where } \|\partial A(y,v,s,\omega)/\partial y\| \ = k(s,\omega).$$

$$a_1(\omega) = exp\ k(\omega).$$

Analogously from equation (7), we obtain

$$\begin{aligned} |\ u(t,\omega)\ |\ &\leq |\ H^T G^{-1}(t_0,t_1,y,v,\omega)\Big[x_1(\omega) - y\big(t_1,t_0,x_0(\omega)\big) \\ &\qquad - \int_{t_0}^{t_1} \Phi(t_1,t,y,v,\omega) f(y,v,\tau,\omega) d\tau\Big]\ | \\ &\leq \|\ H^T G^{-1}(t_0,t_1,y,v,\omega)\Big[x_1(\omega) - y\big(t_1,t_0,x_0(\omega)\big)\Big]\ \| \\ &\qquad + \|\ H^T G^{-1}(t_0,t_1,y,v,\omega)\ \| \int_{t_0}^{t_1} \|\ \Phi(t_1,t,y,v,\omega)\ \|\ |\ f(y,v,\tau,\omega)\ |\ d\tau \end{aligned}$$

$$i.e., |\ u(t,\omega)\ |\ \leq b_0(\omega) + b_1(\omega) \int_{t_0}^{t_1} g\big(\alpha(\tau,\omega),\beta(\tau,\omega),\tau,\omega\big) d\tau \tag{11}$$

where

$$b_0(\omega) = \|\ H^T G^{-1}(t_0,t_1,y,v,\omega)\big[x_1(\omega) - y\big(t_1,t_0,x_0(\omega)\big)\big]\ \|$$

$$b_1(\omega) = \|\ H^T G^{-1}(t_0,t_1,y,v,\omega)\ \|\|\ \Phi(t_1,t,y,v,\omega)\ \|\ .$$

Here $a_0(\omega)$ depends on the initial value $x_0(\omega)$, and $b_0(\omega)$ depends on both the initial and terminal values $x_0(\omega), x_1(\omega)$. But $a_1(\omega), a_2(\omega), b_1(\omega)$ are constants defined only by the system parameters $k_a(t,\omega), k_b(t,\omega)$ and the control interval J.

Therefore, in order that a subset S defined from equation (9) satisfies the theorem, it is sufficient that the right hand side of equations (10) and (11) be smaller than $\alpha(t,\omega)$ and

$\beta(t,\omega)$.

Theorem 2. *Assume that the system (1) satisfies assumptions (2), (3) and condition (5). Then for system (1) to be completely sample controllable on* $J = [t_0, t_1]$, *it is sufficient that the inequality relations*

$$a_0(\omega) + a_1(\omega)\int_{t_0}^{t} k_b(\tau,\omega)\beta(\tau,\omega)d\tau \\ + a_1(\omega)\int_{t_0}^{t} g\big(\alpha(\tau,\omega),\beta(\tau,\omega),\tau,\omega\big)d\tau \leq \alpha(t,\omega) \tag{12}$$

$$b_0(\omega) + b_1(\omega)\int_{t_0}^{t_1} g\big(\alpha(\tau,\omega),\beta(\tau,\omega),\tau,\omega\big)d\tau \leq \beta(t,\omega) \tag{13}$$

have at least one non-negative solution $\big(\alpha(t,\omega),\beta(t,\omega)\big)$ *for any* $a_0, b_0 \in R[\Omega, R_+^*]$ *and for some* $a_1, b_1 \in R[\Omega, R_+]$ *defined by the system equations. Here* $R_+^* = \{x \in R : x > 0\}$.

This theorem can be simplified when the nonlinear function does not depend on x or u. In equation (1) , if f does not contain u, *i.e.*, $f(\cdot) = f(x,t,\omega)$, and the assumption (3) is changed to

$$|\, f(x,t,\omega)\,| \leq g(\,|\,x\,|,t,\omega), \tag{14}$$

then (12) and (13) are respectively converted to

$$a_0(\omega) + a_1(\omega)\int_{t_0}^{t} k_b(\tau,\omega)\beta(\tau,\omega)d\tau \\ + a_1(\omega)\int_{t_0}^{t} g\big(\alpha(\tau,\omega),\tau,\omega\big)d\tau \;\leq \alpha(t,\omega) \tag{15}$$

$$b_0(\omega) + b_1(\omega)\int_{t_0}^{t_1} g\big(\alpha(\tau,\omega),\tau,\omega\big)d\tau \;\leq \beta(t,\omega). \tag{16}$$

Then, from those inequalities we have

$$\begin{aligned}\alpha'(t,\omega) &\geq a_1(\omega)k_b(t,\omega)\beta(t,\omega) + a_1(\omega)g\big(\alpha(t,\omega),t,\omega\big)\\ &\geq a_1(\omega)k_b(t,\omega)\Big[b_0(\omega) + b_1(\omega)\int_{t_0}^{t_1} g\big(\alpha(s,\omega),s,\omega\big)ds\Big] + a_1(\omega)g\big(\alpha(t,\omega),t,\omega\big)\\ &= c_0(t,\omega) + c_1(t,\omega)\int_{t_0}^{t_1} g\big(\alpha(s,\omega),s,\omega\big)ds + a_2(\omega)g\big(\alpha(t,\omega),t,\omega\big),\end{aligned}$$

where $c_0(t,\omega) = a_1(\omega)k_b(t,\omega)b_0(\omega)$

$c_1(t,\omega) = a_1(\omega)k_b(t,\omega)b_1(\omega)$

and $\alpha(t_0,\omega) \geq a_0(\omega)$.

Theorem 3. *Consider the nonlinear sample control system of the form*

$$x'(t,\omega) = A(x,u,t,\omega) + B(x,u,t,\omega)u(t,\omega) + f(x,t,\omega) \tag{17}$$

which satisfies assumptions (2) and (14) and condition (5). If there exists at least one non-negative solution $\alpha(t,\omega)$ of the inequalities

$$\begin{aligned}&\alpha'(t,\omega) \geq c_0(t,\omega) + c_1(t,\omega)\int_{t_0}^{t_1} g\big(\alpha(\tau,\omega)\tau,\omega\big)d\tau + a_1(\omega)g\big(\alpha(\tau,\omega)\tau,\omega\big)\\ &\alpha(t_0,\omega) \geq a_0(\omega)\end{aligned} \tag{18}$$

for any $c_0 \in IB\big[J, R[\Omega, R_+^*]\big]$, $a_0 \in R[\Omega, R_+^*]$, $a_1 \in R[\Omega, R_+]$ and $c_1 \in IB\big[J, R[\Omega, R_+]\big]$, then the system (17) is completely sample controllable on $J = [t_0, t_1]$.

In case that f in (1) does not contain x, *i.e.* $f(\cdot) = f(u,t,\omega)$, and if assumption (3) changes to

$$|f(u,t,\omega)| \le g(|u|,t,\omega), \tag{19}$$

then (12) and (13) are respectively altered to

$$\begin{aligned} a_0(\omega) + a_1(\omega)\int_{t_0}^{t} k_b(\tau,\omega)\beta(\tau,\omega)d\tau + a_2(\omega)\int_{t_0}^{t} g\big(\beta(\tau,\omega),\tau,\omega\big)d\tau &\le \alpha(t,\omega), \\ b_0(\omega) + b_1(\omega)\int_{t_0}^{t_1} g\big(\beta(\tau,\omega),\tau,\omega\big)d\tau &\le \beta(t,\omega). \end{aligned}$$

But if the second inequality has a solution $\beta(t,\omega)$, then the first inequality always has a solution. Therefore, we have the following.

Theorem 4. *Consider the non-linear sample control system of the form*

$$x'(t,\omega) = A(x,u,t,\omega) + B(x,t,u,\omega)u(t,\omega) + f(u,t,\omega) \tag{20}$$

which satisfies assumptions (2) and (19) and condition (5). If there exists at least one non-negative solution $\beta(t,\omega)$ of the inequality

$$\beta(t,\omega) \ge b_0(\omega) + b_1(\omega)\int_{t_0}^{t_1} g\big(\beta(t,\omega),\tau,\omega\big)d\tau$$

for any $b_0 \in R[\Omega, R_+^*]$ and $b_1 \in R[\Omega, R_+]$, then system (20) is completely sample controllable on $J = [t_0, t_1]$.

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