

THE OSCILLATOR GROUP AS A HOMOGENEOUS SPACETIME

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To our friend, Professor and Academician Radu Rochas-Rosca.

Abstract. *Some results on the oscillator group are reported. Specifically: some of their properties as a causal homogeneous spacetime; the oscillator group as a symmetric Lorentzian space which is solution of sourceless Einstein-Yang-Mills equations; the Lorentzian-homogeneous structures and their associated groups of isometries on the oscillator group with left-invariant Lorentzian metrics; the Lorentzian homogeneous structures associated to a bi-invariant metric; and then the oscillator group as solution of Einstein-Yang-Mills equations with sources.*

1. INTRODUCTION

The oscillator group can be described as a semidirect product of the three-dimensional Heisenberg group H_1 and \mathbb{R} under certain action of \mathbb{R} on H_1 . It was named in this way when introduced by Streater in [15], as its Lie algebra is isomorphic to the one generated by the differential operators associated to the harmonic oscillator problem. This group is interesting from the points of view of both physics and differential geometry (see, e.g., [3,5,6,9,10,11,12,13,15]). For instance, the oscillator group is the only non-commutative four-dimensional simply connected solvable Lie group which has a bi-invariant Lorentzian metric (see [12,13]).

Moreover, it is [10,11] an example of homogeneous spacetime, which as a causal space satisfies the so-called causal continuity, which is a relatively strong property among the hierarchy of causal spacetimes. Levchev studied in [9] the oscillator group with the usual bi-invariant Lorentzian metric, proving that this group (which geometrically is a Lorentzian symmetric space and physically is related to an isotropic electromagnetic field) provides a solution of the sourceless Einstein-Yang-Mills equations (see §3).

On the other hand, the oscillator group can be represented as a (non-necessarily symmetric) homogeneous Lorentzian space in different manners, which can be obtained from reductive pairs associated to Lorentzian homogeneous structures (see §4). Thus the question arises of whether such reductive pairs provide new solutions of the Einstein-Yang-Mills

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equations. We consider the family of left-invariant metrics $\{g_\varepsilon : -1 < \varepsilon < 1\}$ defined by Levichev [11] on the oscillator group, which are not bi-invariant except for $\varepsilon = 0$, and we show that only the metric g_0 (the case considered in [3]) yields new solutions, all of them with sources.

2. THE OSCILLATOR GROUP AS A HOMOGENEOUS SPACETIME

Let \mathfrak{g} be the Lie algebra with generators P, X, Y, Q , and brackets

$$[X, Y] = P, \quad [Q, X] = Y, \quad [Q, Y] = -X.$$

The corresponding simply connected Lie group G is called the *oscillator group*. It may be considered as a semidirect product $H_1 \rtimes \mathbb{R}$, where $H_1 \equiv \mathbb{R} \times \mathbb{C}$ is the three-dimensional Heisenberg group.

$$(p, z, q) \cdot (p', z', q') = (p + p' + \frac{1}{2} \operatorname{Im}(\bar{z}e^{iq}z'), z + e^{iq}z', q + q').$$

One considers on G the family $\{g_\varepsilon : -1 < \varepsilon < 1\}$ of left-invariant Lorentzian metrics given at the identity, with respect to the basis $\{P, X, Y, Q\}$ of \mathfrak{g} , by (see [11])

$$g_\varepsilon = \begin{pmatrix} \varepsilon & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \varepsilon \end{pmatrix}. \quad (1)$$

If $\varepsilon = 0$, the corresponding Lorentzian metric is also right-invariant and hence (G, g_0) is a symmetric space. In the other cases, g_ε is not bi-invariant. With respect to the global coordinate system (p, x, y, q) on G , where $x(p, z, q) = \operatorname{Re}(z)$, $y(p, z, q) = \operatorname{Im}(z)$, the metric g_ε is given by

$$\begin{pmatrix} \varepsilon & \frac{\varepsilon}{2}y & -\frac{\varepsilon}{2}x & 1 \\ \frac{\varepsilon}{2}y & \frac{\varepsilon}{4}y^2 + 1 & \frac{\varepsilon}{4}xy & \frac{1}{2}y \\ -\frac{\varepsilon}{2}x & -\frac{\varepsilon}{4}xy & \frac{\varepsilon}{4}x^2 + 1 & -\frac{1}{2}x \\ 1 & \frac{1}{2}y & -\frac{1}{2}x & \varepsilon \end{pmatrix}.$$

The Levi-Civita connection is defined by

$$2g_\varepsilon(\nabla_U V, W) = g_\varepsilon([U, V], W) - g_\varepsilon([V, W], U) + g_\varepsilon([W, U], V),$$

for all $U, V, W \in \mathfrak{g}$. So, the covariant derivatives between generators are given by

$$\begin{aligned} \nabla_P X &= -\frac{\varepsilon}{2} = \nabla_X P, \quad \nabla_X Q = -\frac{1}{2}Y, \quad \nabla_Q X = \frac{1}{2}Y; \\ \nabla_P Y &= \frac{\varepsilon}{2}X = \nabla_Y P, \quad \nabla_Y Q = \frac{1}{2}X, \quad \nabla_Q Y = -\frac{1}{2}X, \\ \nabla_X Y &= \frac{1}{2}P = -\nabla_Y X. \end{aligned}$$

The nontrivial components of the curvature tensor field, with

$$R_{WU}V = \nabla_{[W,U]}V - \nabla_W \nabla_U V + \nabla_U \nabla_W V,$$

are given by

$$\begin{aligned}
3R_{PX}P &= \frac{\varepsilon^2}{4}X, & R_{PY}P &= \frac{\varepsilon^2}{4}Y, \\
R_{PX}X &= -\frac{\varepsilon}{4}P, & R_{PY}Y &= -\frac{\varepsilon}{4}P, \\
R_{PX}Q &= \frac{\varepsilon}{4}X, & R_{PY}Q &= \frac{\varepsilon}{4}Y, \\
R_{XY}X &= -\frac{3\varepsilon}{4}Y, & R_{XY}Y &= \frac{3\varepsilon}{4}X, \\
R_{XQ}P &= -\frac{\varepsilon}{4}X, & R_{YQ}P &= -\frac{\varepsilon}{4}Y, \\
R_{XQ}X &= \frac{1}{4}P, & R_{YQ}Y &= \frac{1}{4}P, \\
R_{XQ}Q &= -\frac{1}{4}X, & R_{YQ}Q &= -\frac{1}{4}Y.
\end{aligned}$$

The Ricci tensor is given by

$$r(U, V) = \text{tr}(W \mapsto R_{UW}V),$$

and, with respect to the basis $\{P, X, Y, Q\}$, we have

$$r = \begin{pmatrix} \frac{\varepsilon^2}{2} & 0 & 0 & \frac{\varepsilon}{2} \\ 0 & -\frac{\varepsilon}{2} & 0 & 0 \\ 0 & 0 & -\frac{\varepsilon}{2} & 0 \\ \frac{\varepsilon}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

If we put $E = (2 - 2\varepsilon)^{-1/2}(P - Q)$, $F = (2 + 2\varepsilon)^{-1/2}(P + Q)$, then $\{E, X, Y, F\}$ is an orthonormal basis of $(\mathfrak{g}, g_\varepsilon)$. The scalar curvature is

$$s = \text{tr}(r) = -r(E, E) + r(X, X) + r(Y, Y) + r(F, F) = -\frac{\varepsilon}{2},$$

and the Einstein tensor is given by

$$r - \frac{1}{2}s g_\varepsilon = \frac{1}{4} \begin{pmatrix} 3\varepsilon^2 & 0 & 0 & 3\varepsilon \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 3\varepsilon & 0 & 0 & 2 + \varepsilon^2 \end{pmatrix}. \quad (2)$$

Now, if $U \neq 0$ is a null vector in (\mathfrak{g}, g_0) then it is either timelike or spacelike in $(\mathfrak{g}, g_\varepsilon)$ according to $\varepsilon < 0$ or $\varepsilon > 0$, respectively. If $-1 < \varepsilon \leq 0$ then $r(U, U) \geq 0$ for every timelike vector U (and $r(U, U) > 0$ if $\varepsilon < 0$), that is, the timelike convergence condition is satisfied (it holds strictly if $\varepsilon < 0$). The weak energy condition on (G, g_ε) , $-1 < \varepsilon \leq 0$, is also satisfied, that is $(r - \frac{1}{2}s g_\varepsilon)(U, U) \geq 0$ for every timelike vector U . Moreover, (G, g_ε) is geodesically complete (see Levičev, [11]).

3. THE SYMMETRIC LORENTZIAN OSCILLATOR GROUP AS A SOLUTION OF SOURCELESS EINSTEIN-YANG-MILLS EQUATIONS

Levichev studied in [9] the spaces called *GCM*-spaces (*G* for group, *C* for causality, *M* for metric), which are *n*-dimensional Lie groups with a left-invariant Lorentz metric satisfying the following condition: *In a neighbourhood of the identity, the union of all non-spacelike future directed 1-parameter subgroups coincides with the union of all non-spacelike future directed geodesics starting from the identity, and this set is a subgroup.*

So, in the *GCM*-spaces, the group, causal and metric structures are closely related, comparatively with other spaces of general relativity. Moreover, all *GCM*-spaces are conformally flat. On the other hand, if *M* is a *GCM*-space, then either the metric on *M* is bi-invariant or its Lie algebra is a fundamental affine algebra (a Lie algebra isomorphic to that of a subgroup of the group of affine transformations of a finite-dimensional real linear space). For $n = 4$ (see [7]) there exist exactly four Lie algebras corresponding to the case of bi-invariant metric:

abelian (it corresponds to Minkowski space),

the algebra of the oscillator group (it corresponds to the case of the isotropic electromagnetic field (see [10]),

$\mathfrak{o}(3) \oplus \mathbb{R}$ (which corresponds to the cosmological model of Segal, *i.e.*, the static closed universe of Einstein), and

$\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$ (the meaning of the corresponding space is not elucidated; on it the energy conditions are not satisfied).

One knows that each Lie group with a bi-invariant metric is a symmetric space. Then, by the simple connectedness one builds uniquely a symmetric pair (G, H) with canonical decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and a fibration $G(G/H, H)$ on the homogeneous space $M = G/H$. The following result, due to Harnad, Tafel and Shnider (see [8, 16]) is known:

Theorem 1 *If \mathfrak{m} admits a nondegenerate $ad(H)$ -invariant bilinear form B , then the bundle $G(G/H, H)$, with the canonical connection and metric induced by B , satisfies the sourceless gauge field equations.*

In [9], Levichev studied the *GCM* spaces as solutions of the sourceless Einstein-Yang-Mills equations. The Einstein-Yang-Mills equations are given by (see, *e.g.*, [2, p. 134]):

$$r - \frac{1}{2} s g + \Lambda g = \kappa T, \quad (3)$$

where g denotes the metric tensor, r the Ricci tensor and s the scalar curvature of g , Λ and κ the cosmological and gravitational constants, respectively, and T the stress-energy tensor of the gauge field with field strength F , given by

$$T(X, Y) = g^{-1}(i_X F, i_Y F) - \frac{1}{4} g(X, Y) \|F\|^2, \quad (4)$$

i denoting the interior product.

In [9] it is proved, in particular,

Theorem 2 *The symmetric oscillator group (G, g_0) is a solution of the sourceless Einstein-Yang-Mills equations. The cosmological constant is $\Lambda = 0$ and the gravitational constant is $\kappa = 4$. The gauge algebra \mathfrak{h} is two-dimensional and abelian.*

4. LORENTZIAN HOMOGENEOUS STRUCTURES AND ASSOCIATED EINSTEIN-YANG-MILLS EQUATIONS

Let (M, g) be a connected C^∞ pseudo-Riemannian manifold, ∇ the Levi-Civita connection and R the curvature tensor field. In [4] it is defined a *pseudo-Riemannian homogeneous structure* on (M, g) as a tensor field S of type $(1, 2)$ such that the connection $\tilde{\nabla} = \nabla - S$ satisfies the Ambrose-Singer equations (see [1])

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0. \quad (1)$$

If g is a Lorentzian metric, a pseudo-Riemannian homogeneous structure (M, g) will be called a *Lorentzian homogeneous structure*.

On the other hand, (M, g) is said to be a *reductive homogeneous pseudo-Riemannian manifold* if $M = G/H$, where G is a connected Lie group acting transitively and effectively on M as a group of isometries, H is the isotropy group at a point $o \in M$, and the Lie algebra \mathfrak{g} of G may be decomposed into a vector space direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where \mathfrak{h} is the Lie algebra of H and $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$. If G is connected and M is simply connected then H is connected, and the condition $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

In [4] it is proved that if (M, g) is connected, simply connected and geodesically complete then it admits a pseudo-Riemannian homogeneous structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold. In [5] the pseudo-Riemannian homogeneous structures are classified using the representation theory of the pseudo-orthogonal groups, finding, as in Tricerri-Vanhoeke's Riemannian case, eight classes: the null one and the seven classes obtained by direct sum of some of three primitive classes S_i ($i = 1, 2, 3$). For instance, the null class corresponds to pseudo-Riemannian symmetric spaces and the primitive class S_3 corresponds to naturally reductive homogeneous pseudo-Riemannian spaces.

Let S be a pseudo-Riemannian homogeneous structure on a connected, simply connected and geodesically complete pseudo-Riemannian manifold (M, g) . We put $\mathfrak{m} = T_o(M)$, where $o \in M$. Let \tilde{R} be the curvature tensor of the connection $\tilde{\nabla} = \nabla - S$. We can consider the holonomy algebra $\tilde{\mathfrak{h}}$ of $\tilde{\nabla}$ as the Lie subalgebra (of the Lie algebra of antisymmetric endomorphisms $\mathfrak{so}_{1,3}(\mathfrak{m})$ of (\mathfrak{m}, g_o)) generated by the operators \tilde{R}_{ZW} , where $Z, W \in \mathfrak{m}$. Then, a Lie bracket is defined (see [1-4]) in the vector space direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{m}$ by

$$\begin{aligned} [U, V] &= UV - VU, & U, V &\in \tilde{\mathfrak{h}}, \\ [U, Z] &= U(Z), & U &\in \tilde{\mathfrak{h}}, Z \in \mathfrak{m}, \\ [Z, W] &= \tilde{R}_{ZW} + S_ZW - S_WZ, & Z, W &\in \mathfrak{m}. \end{aligned} \quad (2)$$

We say that $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is the *reductive pair* associated to the pseudo-Riemannian homogeneous structure S . Let \tilde{G} be the connected simply connected Lie group whose Lie algebra is $\tilde{\mathfrak{g}}$

and \tilde{H} the connected Lie subgroup of \tilde{G} whose Lie algebra is $\tilde{\mathfrak{h}}$. Then \tilde{G} acts transitively on M as a group of isometries and M is diffeomorphic to \tilde{G}/\tilde{H} . If K is the set of the elements of \tilde{G} which act trivially on M , then K is a discrete normal subgroup of \tilde{G} , and the Lie group $G = \tilde{G}/K$ acts transitively and effectively on M as a group of isometries, with isotropy group $H = \tilde{H}/K$. Then M is diffeomorphic to the reductive homogeneous pseudo-Riemannian manifold G/H .

4.1 THE NON-BI-INVARIANT LORENTZIAN OSCILLATOR GROUP

Let g_ε be the left-invariant metric in 1, and $\{\eta, \alpha, \beta, \xi\}$ the basis dual to $\{P, X, Y, Q\}$. In [6] it is proved

Theorem 3 *All the Lorentzian homogeneous structures on the oscillator group (G, g_ε) , $-1 < \varepsilon < 1$, $\varepsilon \neq 0$, are given by*

$$S_\theta = \frac{\varepsilon}{2}\beta \otimes (\eta \wedge \alpha) - \frac{\varepsilon}{2}\alpha \otimes (\eta \wedge \beta) - \frac{1}{2}\beta \otimes (\alpha \wedge \xi) + \frac{1}{2}\alpha \otimes (\beta \wedge \xi) + \theta \otimes (\alpha \wedge \beta), \quad (3)$$

where $\theta = a\eta + b\xi$, $a, b \in \mathbb{R}$.

As a consequence, we note that if $\varepsilon \neq 0$, (G, g_ε) does not admit the Lorentzian homogeneous structure $S = 0$ and thus it is not a symmetric Lorentzian space. If $\theta = \frac{\varepsilon}{2}\eta + \frac{1}{2}\xi$ then S_θ is of type S_3 and hence (G, g_ε) , $\varepsilon \neq 0$, is a naturally reductive Lorentzian space.

Consider now the reductive pair $(\tilde{\mathfrak{g}}_\theta, \tilde{\mathfrak{h}}_\theta)$ associated to S_θ , where $\tilde{\mathfrak{h}}_\theta$ is the holonomy algebra of the connection $\tilde{\nabla}_\theta = \nabla - S_\theta$. With respect to the basis $\{P, X, Y, Q\}$ of \mathfrak{g} , the connection $\tilde{\nabla} = \tilde{\nabla}_\theta$ is given by the non-trivial values

$$\tilde{\nabla}_P X = -(\frac{\varepsilon}{2} + a)Y, \quad \tilde{\nabla}_P Y = (\frac{\varepsilon}{2} + a)X, \quad \tilde{\nabla}_Q X = (\frac{1}{2} - b)Y, \quad \tilde{\nabla}_Q Y = (b - \frac{1}{2})X,$$

where $\theta = a\eta + b\xi$. The holonomy algebra $\tilde{\mathfrak{h}}_\theta$ of $\tilde{\nabla}_\theta$ is generated by

$$U = \tilde{R}_{XY} = \left(\frac{\varepsilon}{2} + a\right)(\beta \otimes X - \alpha \otimes Y),$$

and, if $a \neq -\varepsilon/2$, $\tilde{\mathfrak{h}}_\theta$ is one-dimensional; in this case the reductive decomposition associated to the Lorentzian homogeneous structure S_θ is $\tilde{\mathfrak{g}}_\theta = \tilde{\mathfrak{h}}_\theta \oplus \mathfrak{g} = \langle\langle U, P, X, Y, Q \rangle\rangle$ and the structure equations, given by (2), where $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_\theta$ and $\mathfrak{m} = \mathfrak{g}$, are

$$\begin{aligned} [P, X] &= [X, U] = (\frac{\varepsilon}{2} + a)Y, & [P, Y] &= [Y, U] = -(\frac{\varepsilon}{2} + a)X, \\ [X, Y] &= U + P, & [Q, X] &= (b + \frac{1}{2})Y, & [Q, Y] &= -(b + \frac{1}{2})X. \end{aligned} \quad (4)$$

Since the manifolds (G, g_ε) are geodesically complete, with each Lorentzian homogeneous structure S_θ on (G, g_ε) , $\varepsilon \neq 0$, given by (3), there are associated a group of isometries \tilde{G}_θ acting transitively and effectively on G . In [6] it is proved that if $\theta = -\frac{\varepsilon}{2}\eta - \frac{1}{2}\xi$, \tilde{G}_θ is the direct product of the three-dimensional Heisenberg group and \mathbb{R} and, if $\theta = -\frac{\varepsilon}{2}\eta + b\xi$, $b \neq -1/2$, \tilde{G}_θ is isomorphic to G . If $\theta = a\eta + b\xi$, $a \neq -\varepsilon/2$, the corresponding group of

isometries is $\tilde{G}_\theta = \mathbb{C} \times \mathbb{R}^2 \times \mathbb{S}^1$ with the operation defined by

$$\begin{aligned} (z, p, q, e^{iu})(z', p', q', e^{iu'}) &= (z + \exp(i((\frac{\varepsilon}{2} + a)p + (b + \frac{1}{2})q + u)) z', \\ & p + p' + \frac{1}{2} \operatorname{Im}(\bar{z} \exp(i((\frac{\varepsilon}{2} + a)p + (b + \frac{1}{2})q + u)) z'), q + q', \\ & \exp(i(u + u' - \frac{\varepsilon + 2a}{4} \operatorname{Im}(\bar{z} \exp(i((\frac{\varepsilon}{2} + a)p + (b + \frac{1}{2})q + u)) z'))), \end{aligned}$$

which acts transitively and effectively on G by

$$\begin{aligned} (z, p, q, e^{iu}) \cdot (p', z', q') &= (p + p' + \frac{1}{2} \operatorname{Im}(\bar{z} \exp(i((\frac{\varepsilon}{2} + a)p + (b + \frac{1}{2})q + u)) z'), \\ & z + \exp(i((\frac{\varepsilon}{2} + a)p + (b + \frac{1}{2})q + u)) z', q + q'), z, z' \in \mathbb{C}, p, q, p', q', u \in \mathbb{R} \end{aligned}$$

As we noted above, if $a \neq -\frac{\varepsilon}{2}$ then $\tilde{\mathfrak{h}}_\theta$ is non-trivial. In this case, we consider the canonical connection ω defined on $\tilde{\mathfrak{g}}_\theta$, which is the $\tilde{\mathfrak{h}}_\theta$ -component of the canonical 1-form on $\tilde{\mathfrak{g}}_\theta$. Since the only nonvanishing $\tilde{\mathfrak{h}}_\theta$ -component of a bracket in (4) is $[X, Y]_{\tilde{\mathfrak{h}}_\theta} = U$, the curvature $F = D\omega$ is determined by $F(X, Y) = -U$, so $F = -(\alpha \wedge \beta) \otimes U$. Then, if \star denotes the Hodge operator with respect to the Lorentzian metric g_ε ,

$$\star F = -\sqrt{1 - \varepsilon^2} (\eta \wedge \xi) \otimes U.$$

From the second Yang-Mills equation $J = D\star F$, where J denotes the current, one obtains

$$J = \sqrt{1 - \varepsilon^2} (\alpha \wedge \beta \wedge \xi) \otimes U.$$

We consider the metric on $\tilde{\mathfrak{g}}_\theta$ which is the direct product of the metric g_ε on \mathfrak{g} and the metric of $\tilde{\mathfrak{h}}$ for which the basic vector U is unitary. Then $\|F\|^2 = 2$ and the stress-energy tensor T , given by (4), is expressed in terms of the basis $\{P, X, Y, Q\}$ of \mathfrak{g} by

$$T = \begin{pmatrix} -\frac{\varepsilon}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{\varepsilon}{2} \end{pmatrix}.$$

From (1), (2) and the above expression for T , it follows that equation (3) is not satisfied for any values of Λ and κ .

Then, for each $\varepsilon \neq 0$, $-1 < \varepsilon < 1$, the non-bi-invariant oscillator group (G, g_ε) does not provide solutions of the Einstein-Yang-Mills equations.

4.2 THE BI-INVARIANT CASE

If $\varepsilon = 0$ in (1), we have the bi-invariant Lorentzian metric $g = g_0$ on the oscillator group G . Given the dual basis $\{\eta, \alpha, \beta, \xi\}$ to $\{P, X, Y, Q\}$, we have (see [3, 6])

Proposition 1 *The Lorentzian homogeneous structures on (G, g) are given by*

$$S = \theta \odot (\alpha \wedge \beta) + \rho \odot (\alpha \wedge \xi) + \sigma \odot (\beta \wedge \xi), \quad (5)$$

where ρ , σ and θ are left-invariant 1-forms on G satisfying the conditions

$$\begin{aligned}\tilde{\nabla}\theta &= 0, \\ \tilde{\nabla}\rho &= \sigma \wedge \theta - \frac{1}{2}\alpha \otimes \theta + \frac{1}{2}\xi \otimes \sigma, \\ \tilde{\nabla}\sigma &= -\rho \wedge \theta - \frac{1}{2}\beta \otimes \theta - \frac{1}{2}\xi \otimes \rho,\end{aligned}\tag{6}$$

with $\tilde{\nabla} = \nabla - S$.

After some long but straightforward calculations, one obtains several families of solutions of equations (6). Among them, the following four families are those which yield the homogeneous structures on (G, g) with associated reductive decompositions $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{m}$ where the holonomy algebra \mathfrak{h} is non-trivial (see [3]).

$$\begin{aligned}\text{(I)} \quad \theta &= x\eta + xy\alpha + xz\beta + (w + \frac{1}{2})\xi, \\ \rho &= -xz\eta - xyz\alpha - (xz^2 + \frac{1}{2})\beta - zw\xi, \\ \sigma &= xy\eta + (xy^2 + \frac{1}{2})\alpha + xyz\beta + yw\xi,\end{aligned}$$

where $x, y, z, w \in \mathbb{R}$, $x \neq 0$, and y, z, w are not all null.

$$\text{(II)} \quad \theta = \frac{1}{2}\xi, \quad \rho = -\frac{1}{2}\beta, \quad \sigma = q\alpha + c\xi,$$

where $q, c \in \mathbb{R}$, $q \neq \frac{1}{2}$.

$$\begin{aligned}\text{(III)} \quad \theta &= \frac{1}{2}\xi, \quad \rho = k\alpha + (t - \frac{1}{2})\beta + b\xi, \\ \sigma &= (\frac{1}{2} - (k^2/t))\alpha - k\beta - (kb/t)\xi,\end{aligned}$$

where $k, t, b \in \mathbb{R}$, $t \neq 0$.

$$\text{(IV)} \quad \theta = a\xi, \quad \rho = -a\beta, \quad \sigma = a\alpha,$$

where $a \in \mathbb{R}$, $a \neq \frac{1}{2}, -\frac{1}{2}$.

Each Lorentzian homogeneous structure S given by (5) defines a connection $\tilde{\nabla} = \nabla - S$, and the curvature operators \tilde{R}_{ZW} , ($Z, W \in \mathfrak{g}$), generate the holonomy algebra $\tilde{\mathfrak{h}}$ of $\tilde{\nabla}$, which is one-dimensional in cases (I), (II) and (III) and two-dimensional and abelian in case (IV). We consider the generators of \mathfrak{h} defined by

$$A = \begin{pmatrix} 0 & z & -y & 0 \\ 0 & 0 & 1 & -z \\ 0 & -1 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

in case (I), $B = (q - \frac{1}{2})^{-1} \tilde{R}_{QY}$ in case (II), $C = \tilde{R}_{QX}$ in case (III), and $U = (a^2 - \frac{1}{4})^{-1} \tilde{R}_{QX}$ and $V = (a^2 - \frac{1}{4})^{-1} \tilde{R}_{QY}$ in case (IV).

One has (see [3]) the reductive pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ associated to each homogeneous structure, where the Lie bracket of $\tilde{\mathfrak{g}}$ is defined by (2), with $\mathfrak{m} = \mathfrak{g}$. In particular, if $a = 0$ in case (IV), one obtains $S = 0$ and the associated reductive pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is a symmetric pair, which defines the oscillator group as a symmetric Lorentzian space; in this case, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{g} = \langle \{U, V, P, X, Y, Q\} \rangle$, with structure equations

$$\begin{aligned}2[U, X] &= P, & [U, Q] &= -X, & [V, Y] &= P, & [V, Q] &= -Y, \\ [Q, X] &= -\frac{1}{4}U, & [Q, Y] &= -\frac{1}{4}V.\end{aligned}$$

For each reductive decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{m}$ corresponding to each one of the four cases above, we can consider the canonical connection ω defined on $\tilde{\mathfrak{g}}$. So we obtain the curvature form $F = D\omega$, and the current $J = D \star F$ in each case, which vanishes only in case (IV) for $a = 0$ (the symmetric case).

On the other hand, we consider a metric on $\tilde{\mathfrak{g}}$ which is the direct product of the metric g_0 on \mathfrak{g} and the metric on $\tilde{\mathfrak{h}}$ for which the basic vectors generating the holonomy are unitary. Then the stress-energy tensor T can be computed, and expressed in terms of the basis $\{P, X, Y, Q\}$ of \mathfrak{g} . It is then a matter of calculation to see that the field equations 3 are satisfied in cases (II), (III) and (IV), and in case (I) if $w = 0$ and $y^2 + z^2 \neq 0$, and to obtain the corresponding cosmological and gravitational constants. One has ([3])

Theorem 4 *The Lorentzian reductive pairs $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ determined by the Lorentzian homogeneous structures on the oscillator group, in cases (I) (with $w = 0$ and $y^2 + z^2 \neq 0$), (II), (III) and (IV) above, provide solutions of the Einstein-Yang-Mills equations for an electromagnetic gauge field with cosmological constant $\Lambda = 0$, and gravitational constant κ given, respectively, by*

$$\frac{1}{2x^2(y^2 + z^2)}, \quad \frac{1}{2(q - \frac{1}{2})^2}, \quad \frac{t^2}{2(k^2 + t^2)}, \quad \frac{1}{4(a^2 - \frac{1}{4})^2}.$$

The gauge algebra $\tilde{\mathfrak{h}}$ is abelian and two-dimensional in case (IV) and one-dimensional in the other three cases. The currents are given by

	J
(I)	$(x_2 \eta \wedge \alpha \wedge \xi - xy \eta \wedge \beta \wedge \xi + w \alpha \wedge \beta \wedge \xi) \odot A$
(II)	$(q - \frac{1}{2})(\eta \wedge \beta \wedge \xi + c \alpha \wedge \beta \wedge \xi) \odot B$
(III)	$(\eta \wedge \alpha \wedge \xi - (k/t)\eta \wedge \beta \wedge \xi + b(1 + (k/t)^2)\alpha \wedge \beta \wedge \xi) \odot C$
(IV)	$2a(a^2 - \frac{1}{4})((\eta \wedge \alpha \wedge \xi) \odot U + (\eta \wedge \beta \wedge \xi) \odot V)$

They are nonvanishing except in case (IV) for $a = 0$.

Remark 1 *If $a = 0$ in case (IV), the associated symmetric pair $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ furnishes a solution of the Einstein-Yang-Mills equations for a sourceless gauge field with cosmological constant $\Lambda = 0$ and gravitational constant $\kappa = 4$ (see Theorem 2).*

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