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ON THE RIEMANNIAN MANIFOLDS WITH BIRECURRENT RICCI TENSOR

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Abstract. *We show that a Riemannian manifold satisfying the condition $RQ = 0$ is Ricci parallel or has degenerate Ricci tensor.*

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1. INTRODUCTION

The recurrent (Riemannian) manifolds were introduced as a natural extension of locally symmetric spaces, by imposing the existence of an exterior 1-form ω such that the covariant derivative of the curvature tensor writes $\nabla_X R = \omega(X)R$. Significant results concerning the recurrent (and the Ricci-recurrent) manifolds have been obtained by E.M. Patterson [15], H.S. Ruse [23], A.G.Walker [30], J.T. Wilmore [22] (see also [1], [3], [4], [6]-[13], [17]-[21]).

The notion was generalized by A. Lichnerowicz, who defined and studied the second-order recurrent (i.e. birecurrect) spaces, i.e. those Riemannian manifolds which admit a (0,2)-type tensor field B, such that

$$\nabla_X \nabla_Y R = B(X,Y)R \tag{1}$$

for every vector fields X and Y.

We call Ricci-birrecurent a Riemannian manifold which admits a tensor field B of (0,2)-type, such that the Ricci tensor Q satisfies

$$\nabla_X \nabla_Y Q = B(X,Y)Q \tag{2}$$

for every vector fields X and Y. Obviously, (1) implies (2).

In [7], A. Lichnerowicz proved that every compact, second-order recurrent manifold, with non-vanishing scalar curvature is locally symmetric, or is locally reducible to the product of a two-dimensional Riemannian manifold with a locally Euclidean manifold. (and, hence, it is recurrent).

The purpose of this paper is to prove an analogous result, for the Ricci-birecurrent manifolds.

Theorem 1 *Let (M,g) be a n-dimensional Riemannian manifold ($n \geq 2$), with birecurrent Ricci tensor. Then M is Ricci-parallel or has degenerate Ricci tensor.*

2. PROOF OF THE MAIN RESULT

First step : Ricci-birecurrent implies $RQ = 0$.

Let (M , g) be a n-dimensional Riemannian manifold ($n \geq 2$), with curvature tensor R , given by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Consider the (0,2)-Ricci tensor Q and the associated Ricci tensor r , of type (1,1), defined by $g(rX,Y) = Q(X,Y)$.

By using the identity

$$(\nabla_X \nabla_Y Q)(Z,W) - (\nabla_Y \nabla_X Q)(Z,W) = Q(R(Z,W)X,Y) + Q(R(Z,W)Y,X)$$

we obtain the symmetry of the tensor field B. In particular, this property implies that the manifold satisfies the relation $RQ = 0$. This means that, for every vector fields X and Y on M, we have $R(X,Y)Q = 0$, with R acting as a derivation on the tensorial algebra.

Second step : $RQ = 0$ and non-degenerate Q implies local product.

First we prove the

Lemma. Let M be a Riemannian manifold with non-degenerate Ricci tensor, satisfying $RQ = 0$. Then, locally, M is a product of two-dimensional Riemannian manifolds with Einstein manifolds.

Proof. In local coordinates, the condition $RQ = 0$ becomes

(3) $$R^s_{jkl} Q_{si} + R^s_{ikl} Q_{sj} = 0$$

where R^s_{jkl} and Q_{ij} are the coordinates of R and Q respectively. The relation (3) is equivalent, in invariant formalism, with any of the two following identities

(4) $$R(X,Y)r = rR(X,Y) \quad , \quad R(rX,Y) = R(X,rY)$$

for every vector fields X , Y on M.

Consider now the local de Rham decomposition of M. Each irreducible factor inherits the (equivalent) relations (4), so has the corresponding Ricci tensor proportional with the projected Riemannian metric. If the factor has dimension different from two, then it is an Einstein manifold. □ (lemma)

So, if Q is nondegenerate then, locally, M decomposes as a product of two-dimensional Riemannian manifolds and/or Einstein manifolds.

Third step : final argument.

If $\nabla^2 Q = 0$, then, by a result of Nomizu and Ozeki [12] (see also Tanno [23]) it follows $\nabla Q = 0$.

Suppose Q is non-degenerate and non- parallel.

Suppose M has a 2-dimensional irreducible factor N, corresponding to a non-vanishing proper function λ of Q. Then the N-projection of Q is the Ricci tensor of N, and it is birrecurrent with the birecurrency tensor $\nabla d\lambda$. Due to the fact that the last tensor is non-null, we deduce that $d\lambda$ is non-parallel (by abuse, we say " λ is non-parallel"). The manifold M cannot have two such 2-dimensional factors, because this would provide two different birecurrency tensors B in (2).

(Another argument: $d\lambda$ cannot be parallel due to the irreducibility of N).

If M has a factor which is (a non Ricci-flat) Einstein manifold, then on the corresponding "projection" the second covariant derivative of the Ricci tensor vanishes, producing a contradiction in (2).

Hence (locally) the only possibility for M is to have at most two factors : a two-dimensional one (with necessarily "non-parallel" scalar curvature) and a Ricci-flat manifold. ■

3. COMMENTS

(i) If moreover, (M, g) is complete and simply connected, then the global de Rham theorem shows that the theorem holds true globally.

(ii) For $n = 3$, the condition $RQ = 0$ is equivalent to semi-symmetry (i.e. $RR = 0$). The decomposition theorem we proved can then be refined, using the classification result from [25].

(iii) Obviously, the "converse" of the theorem is true: a Ricci-parallel Riemannian manifold (i.e. a product of Einstein manifolds) and the product of a two-dimensional Riemannian manifold with a Ricci-flat one are Ricci-birecurrent manifolds.

(iv) Many remarkable tensor fields are parallel, provided their k-th covariant derivative (for some k) vanishes ([13], [26]). Does this reduction property hold also for (higher order) recurrency ? A theorem of Lichnerowicz [7] states that a compact Riemannian manifold with birecurrent curvature ("second order recurrent space") has recurrent curvature. Our theorem gives another reason to belive that, in general, higher order covariant equations (involving curvature tensors) reduce to lower order ones or lead to degenerate tensors.

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