LIBERTAS MATHEMATICA, VOL. XIX (1999)

# GENERALIZED FERMI-WALKER TRANSPORT

**G. PRIPOAE**

**Abstract.** *We extend the Fermi-Walker parallel transport in General Relativity, requiring only spatial directions conservation. Examples of corresponding Thomas-like precessions in Schwarzschild geometry are given. The constructions are similar to the extension of Schouten's and Vranceanu's connections from non-holonomic geometrization.*

**1991 Mathematics Subject Classification** : 53C50, 83C10, 83C99

## 1. INTRODUCTION

The choice of an appropriate reference frame is a fundamental and controversial problem in Astronomy ([1]): one needs a"center" and several "fixed" directions. In a general relativistic setting, a freely falling observer $\gamma$ may carry gyroscopes to focus toward distant "fixed" stars (each transported by Levi-Civita parallelism along $\gamma$). If $\gamma$ is not freely falling, the Levi-Civita parallelism do not preserve $\gamma$- restspaces anymore. In this case, in order to define "constant" directions, another parallelism is used: the Fermi-Walker transport ([8], [9]). This parallelism is an isometry between tangent spaces along $\gamma$.

In this paper we extend the Fermi-Walker transport, by requiring only spatial directions conservation. In §2 we characterize this family of induced covariant derivatives along an accelerated observer $\gamma$. If we impose also angle (or norm) conservation, we obtain another remarquable class of parallel transports; between them, we recover some particular connections studied in the special relativistic framework (Hehl, Lemke, Mielke [2]). In §3 we consider the relevance of the new parallel transport for the Schwarzschild geometry. Finally, we point out several similarities between the Fermi-Walker transport and Schouten's and Vranceanu's connections from non-holonomic geometrization.

## 2. GENERALIZED FERMI-WALKER (INDUCED) CONNECTIONS

Consider a $n$- dimensional Lorentzian manifold $(M, g)$, with signature convention (-, +, ...,+). Let $\xi \in \mathcal{X}(M)$ a fixed temporal vector field, giving a time orientation on $M$. The triple $(M, g, \xi)$ is a spacetime (usually, $n = 4$). An *observer* is a proper time parametrized curve $\gamma : I \to M$, with all its velocity vectors temporal and future oriented, i.e.

$$g(\gamma', \gamma') = -1 \quad , \quad g(\xi \circ \gamma, \gamma') < 0$$

Denote by $\nabla$ the Levi-Civita connection of $g$ and by $\mathcal{X}_\gamma$ the module of vector fields along $\gamma$. The restspace of $\gamma$ in $\gamma(t)$ is the orthogonal complement of $\gamma'(t)$ in the tangent space $T_{\gamma(t)}M$ (and obviously is a spatial subspace).

Consider $C_\gamma$ the set of all induced linear connections along $\gamma$; each such connection defines a parallel transport between the tangent spaces of $M$ in every pair of $\gamma$-points.

**Definition 1** *We call generalized Fermi- Walker connection along $\gamma$ an induced linear connection $\tilde{\nabla} \in C_\gamma$ whose parallel transport conserves the restspaces of $\gamma$. Such a transport is called spatial conformal ( or spatial isometric) if moreover it preserves angles between (resp. norms of) restspace directions along $\gamma$.*

**Remark 1** *(i) The classical Fermi- Walker connection is defined ([8], [9]) by*

$$\nabla^0_{\gamma'}X = \nabla_{\gamma'}X + g(\gamma', X)\nabla_{\gamma'}\gamma' - g(\nabla_{\gamma'}\gamma', X)\gamma' \tag{1}$$

*for every vector field $X \in \mathcal{X}_\gamma$. It is known that the corresponding parallel transport is a spatial isometry and that $\gamma$ is autoparallel with respect to $\nabla^0$.*

*(ii) If a given non-vanishing vector field $X$ along $\gamma$, orthogonal to $\gamma$, is $\tilde{\nabla}$-parallel, we say it has constant direction (in the restspace of $\gamma$). In the opposite case, the direction changes: this is the so-called Thomas precession, measured by $\tilde{\nabla}_\gamma X$.*

*(iii) The Fermi-Walker connection is relevant for accelerating observers only. (Connections $\nabla$ and $\nabla^0$ coincide along $\gamma$ if and only if $\gamma$ is a freely falling observer, that is $\gamma$ is a geodesic.) In exchange, the generalized Fermi-Walker connection is interesting for both cases.*

**Proposition 1** *Let $\tilde{\nabla}$ be a generalized Fermi-Walker connection along $\gamma$. Then there exists a unique (1,2)-tensor field $A$ along $\gamma$ such that*
$(2) and g(A(\gamma', X), \gamma') = 0 (3) for every vector field$

$$\tilde{\nabla}_{\gamma'}X = \nabla_{\gamma'}X + g(\gamma', X)\nabla_{\gamma'}\gamma' - g(\nabla_{\gamma'}\gamma', X)\gamma' + A(\gamma', X)$$

*$X \in \mathcal{X}_\gamma$, orthogonal to $\gamma$.*

**Proof.** Define $A = \tilde{\nabla} - \nabla^0$ along $\gamma$. If $X \in \mathcal{X}_\gamma$ is orthogonal to $\gamma$, then $\tilde{\nabla}_{\gamma'}X$ is orthogonal to $\gamma$ if and only if relation (3) is satisfied. ■

**Corollary 1** *Consider $\tilde{\nabla}$ a generalized Fermi-Walker connection, given by (2)+(3). Then*

*(i) $\gamma$ is $\tilde{\nabla}$-autoparallel if and only if $A(\gamma', \gamma') = 0$.*

*(ii) The transport of $\tilde{\nabla}$ is spatial conformal if and only if*

$$\{g(X, A(\gamma', X))g(Y, Y) + g(Y, A(\gamma', Y))g(X, X)\}g(X, Y) =$$

$$= \{g(A(\gamma', X), Y) + g(A(\gamma', Y), X)\}g(X, X)g(Y, Y) \tag{4}$$

*for every $X, Y \in \mathcal{X}_\gamma$, orthogonal to $\gamma$.*

*(iii) The following statements are equivalent: (a) the transport of $\tilde{\nabla}$ is spatial isometric; (b) $\tilde{\nabla}_\gamma g = 0$ on all restspace directions; (c) the linear operator $A(\gamma', .)$ is skew-adjoint.*

*(iv) If the parallel transport of $\tilde{\nabla}$ is an isometry, then each of the previous properties are satisfied.*

**Proof.** By direct computation, starting with the relation (2) and eliminating $\nabla$ and $\tilde{\nabla}$. ■

**Remark 2** *(i) Condition (4) is strictly weaker than the condition required for spatial isometry. Indeed, we may choose $A(\gamma', .)$ as an appropriate linear operator acting on each restspace, with real eigenvalues. This operator is not skew-adjoint, except for the trivial case.*

*The same kind of argument shows that the converse of property (iv) is false, in general.*

*(ii) In a special relativistic framework, Hehl, Lemke and Mielke ([2]) also considered extensions of the Fermi-Walker transport; ab initio, they supposed that the transport was spatial isometric.*

## 3. APPLICATION TO SCHWARZSCHILD GEOMETRY

Consider the exterior Schwarzschild spacetime $N = \mathbf{R} \times (2M, \infty) \times S^2$ , where $S^2$ is the unit sphere and $M$ is the mass of the central star; denote $(t, r, \theta, \varphi)$ the canonical coordinates on $N$. The line element on $N$ is a warped product:

$$ds^2 = -hdt^2 + h^{-1}dr^2 + r^2(d\theta^2 + cos^2\theta d\varphi^2)$$

where $h(r) = 1 - \frac{2M}{r}$.

The time orientation on $N$ is given by the vector field $U = (1/\sqrt{h})\partial_t$, whose integral curves are the Schwarzschild observers. These are accelerating ones, due to the fact that

$$\nabla_U U = \frac{M}{r^2}\partial_r$$

Hence, the parallel transport along a Schwarzschild observer $\alpha$ does not conserve $\alpha$-restspaces.

The general form of a generalized Fermi-Walker connection along $\alpha$ is

$$\tilde{\nabla}_{\alpha'}\partial_t = A(\alpha', \partial_t) \quad , \quad \tilde{\nabla}_{\alpha'}\partial_r = A(\alpha', \partial_r) \quad , \quad \tilde{\nabla}_{\alpha'}X = A(\alpha', X)$$

where $X$ is the lift of a vector field from $S^2$ and $A$ is a (1,2)-tensor field on $N$ satisfying

$$g(A(\alpha', \partial_r), \partial_t) = g(A(\alpha', X), \partial_t) = 0$$

(All expressions are restricted along $\alpha$). When $A$ vanishes, we recover the classical Fermi-Walker transport, which -in particular- paralelizates the canonical basis $\partial_r$, $\partial_\theta$, $\partial_\varphi$.

**Example 1** *We may choose $A$ satisfying*

$$A(\alpha', \partial_t) = A(\alpha', \partial_\varphi) = 0 \quad , \quad A(\alpha', \partial_r) = \partial_\theta \quad , \quad A(\alpha', \partial_\theta) = -hr^2 \partial_r$$

*Then, as we may check also directly, the generalized Fermi-Walker transport is a spatial isometry. The direction $\partial_\varphi$ remains parallel, but $\partial_r$ and $\partial_\theta$ do not.*

## 4. COMMENTS

(i) We may wonder to what extent the generalized Fermi-Walker transport offers a better choice of reference systems. In many astrophysical records, distant star positions are considered "fixed" directions. But are the respective directions parallel with respect to some connection (eventually a classical Fermi-Walker one) ?

Let $\gamma$ an accelerating observer. If there exist three independent directions in his restspaces, parallel with respect to $\nabla^0$, we may remain in the classical setting.

If not, the observer will choose three (remarquable) independent directions in his restspace; then, there exists a (non-unique) generalized Fermi-Walker connection $\tilde{\nabla}$ which parallelizates the respective directions. Three giroscopes put on the three directions will describe the invariance or the precession of any other direction. In case there exist several such connections, they may provide data for average measurements.

Consider now another accelerating observer $\alpha$. The classical setting forces him to use the same Fermi-Walker connection $\nabla^0$, violating - at least the spirit of - the Principle of Relativity. It may happen that the three "fixed (i.e. $\nabla^0$ -parallel) stars" for $\gamma$ remain no longer fixed for $\alpha$. If generalized Fermi-Walker connections are considered, the observers are allowed to use the same frame directions, but with different tensor fields $A$ (i.e. with different parallelisms).

(ii) The construction in Proposition 2 is similar to the construction of Schouten's and Vranceanu's connections on arbitrary almost product manifolds ([4], [5], [6]). Consider a differentiable manifold $M$ endowed with a field of endomorphisms $P \in T_1^1(M)$, i.e. $P^2 = Id$. This gives rise to a pair of complementary distributions $\mathcal{D}$ and $\mathcal{D}'$, with corresponding projectors $V$, resp. $V'$.

Let $\nabla$ be a linear connection on $M$. The generalized Schouten's and Vranceanu's connections write

$$2\nabla^S_X Y = 2\nabla_X Y + P(\nabla_X P)Y + B(X,Y)$$

and

$$4\nabla^V_X Y = 4\nabla_X Y + (\nabla_{PY} P)X + P(\nabla_Y P)X + 2P(\nabla_X P)Y + C(X,Y)$$

where $B$ and $C$ are (1,2)-tensor fields satisfying

$$B(X,Y) = PB(X,PY) \quad , \quad C(X,Y) - PC(X,PY) = C(PX,PY) - PC(PX,Y)$$

These connections contain much information concerning the structure of the manifold, and allow to consider various kinds of exotic parallelism. Their invariants (torsion, curvature) were studied as an attempt to geometrize non-holonomic systems.

When we have an observer $\gamma$ on a spacetime, then the two complementary distributions are replaced - formally - by the tangent direction and the restspace of $\gamma$. So, by analogy, we may define the (generalized) Schouten's and Vranceanu's (induced) connections along $\gamma$; from the (classical) Schouten's connection we derive the (classical) Fermi-Walker connection. It is very plausible that the general (Schouten-Vranceanu) case will provide new (induced) connections of Fermi-Walker type.

## REFERENCES

[1] N. Capitaine, B. Guinot - *Systèmes de référence en astronomie*, *C.R. Acad. Sci. Paris* **324** (1997), sér. II, 725-738

[2] F.W. Hehl, J. Lemke, E.W. Mielke - *Two lectures on fermions and gravity*, in *Geometry and Theoretical Physics*, J. Debrus and A.C. Hirshfeld (eds.), Springer Verlag, 1991

[3] B. O'Neill - *Semi-Riemannian Geometry* , Academic Press, New York, 1983

[4] G.T. Pripoae - *Une generalisation des connexions de Schouten et de Vranceanu*, *Proc. Rom. Nat. Colloq. Geom. Top., Timisoara 1985*, 223-228

[5] G.T. Pripoae - *Connexions de Schouten et de Vranceanu sur des f-varietes*, *Rev. Mat. Univ. Parma*, **12** (1986), 195-201

[6] G.T. Pripoae - *Sur les connexions de Schouten et de Vranceanu*, *Rev. Roum. Math. Pures Appl.*, **33** (1988), 447-456

[7] R. Rosca , Gh. Vranceanu - *Introduction in Relativity and pseudo-Riemannian geometry*, Ed. Academiei, Bucharest, 1976

[8] R.K. Sachs, H. Wu - *General Relativity for Mathematicians*, Springer Verlag, New York, 1977

[9] S. Weinberg - *Gravitation and Cosmology*, J. Wiley Publ., New York, 1972