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TIME PERIODIC SOLUTIONS FOR A HIGHER ORDER NONLINEAR HYPERBOLIC SYSTEM

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Abstract. *We study the existence of time periodic solutions to a class of hyperbolic partial differential systems.*

1. INTRODUCTION

In this paper we shall investigate the existence of time periodic solutions for the following nonlinear system:

$$(S) \qquad \begin{cases} \dfrac{\partial i}{\partial t} + \displaystyle\sum_{k=0}^{n} a_k(x) \frac{\partial^k v}{\partial x^k} + \alpha(x,i) = f(t,x) \\[2ex] \dfrac{\partial v}{\partial t} - \displaystyle\sum_{k=0}^{n} (-1)^k \frac{\partial^k}{\partial x^k}[a_k(x) i] + \beta(x,v) = g(t,x), \end{cases}$$

$$0 < x < 1, \quad t > 0$$

with the boundary condition:

$$\begin{pmatrix} col\,((L_1 i)(t,0),\ -(L_1 i)(t,1), \ldots,\ (L_n i)(t,0),\ -(L_n i)(t,1)) \\ col\,(s_1 w_1'(t), \ldots,\ s_m w_m'(t)) \end{pmatrix} \in$$

(BC)

$$\in -G \begin{pmatrix} col\,(v(t,0),\ v(t,1), \ldots,\ \dfrac{\partial^{n-1} v}{\partial x^{n-1}}(t,0),\ \dfrac{\partial^{n-1} v}{\partial x^{n-1}}(t,1)) \\ col\,(w_1(t), \ldots,\ w_m(t)) \end{pmatrix} + B(t), \quad t > 0,$$

where $L_j i = \displaystyle\sum_{k=j}^{n} (-1)^{k-j} \frac{\partial^{k-j}}{\partial x^{k-j}}[a_k(x) i]$, $j = \overline{1,n}$, $(n \in \mathbb{N}^*)$, $B(t) = col(b_1(t), b_2(t), \ldots, b_{2n+m}(t))$.

This problem has applications in the theory of integrated circuits and hydraulics (for $n = 1$) and in the elastic beam theory (for $n = 2$), (see [5,9,10] for futher references). The existence, uniqueness and some regularity properties of the strong and weak solutions to the problem (S), (BC) with the initial data:

$$(IC) \qquad \begin{cases} i(0,x) = i_0(x), \quad v(0,x) = v_0(x), \ 0 < x < 1, \\ w_j(0) = w_{j_0}, \ j = \overline{1,m} \end{cases}$$

have been investigated in [5,7]. This problem is a generalization of the one who we studied in [6] (where $n = 1$) and it also generalizes the problem studied in [11] (where no function $w_j(t)$ appears, $j = \overline{1,m}$ and $B(t) \equiv 0$ in (BC)). For the basic concepts and results in the theory of monotone operators and nonlinear evolution equations of monotone type we refer the reader to [1].

We introduce the assumptions that we shall use in this paper:

- (H1) The functions $a_k \in W^{k,\infty}(0,1)$, $k = \overline{0,n}$, $a_n(x) \neq 0$, $(\forall) x \in [0,1]$.
- (H2) a) The functions $x \to \alpha(x,p)$ and $x \to \beta(x,p)$ are in $L^2(0,1)$, for any $p \in \mathbb{R}$. Besides, the functions $p \to \alpha(x,p)$ and $p \to \beta(x,p)$ are continuous and nondecreasing from $\mathbb{R}$ into $\mathbb{R}$, for a.a. $x \in (0,1)$.

  b) There exist the constants $a, b > 0$ and the functions $c, d \in L^2(0,1;\mathbb{R}_+)$ such that:

$$|\alpha(x,p)| \geq a|p| - c(x), \quad |\beta(x,p)| \geq b|p| - d(x); \quad (\forall)\, p \in \mathbb{R}, \ \text{a.a. } x \in (0,1).$$

- (H3) a) $G : D(G) \subset \mathbb{R}^{2n+m} \to \mathbb{R}^{2n+m}$ is a maximal monotone mapping (possibly multivalued), $D(G) \neq \emptyset$. Moreover, G can be split in:

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

  where $G_{11} : D(G_{11}) \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $G_{12} : D(G_{12}) \subset \mathbb{R}^{m} \to \mathbb{R}^{2n}$,
  $G_{21} : D(G_{21}) \subset \mathbb{R}^{2n} \to \mathbb{R}^{m}$, $G_{22} : D(G_{22}) \subset \mathbb{R}^{m} \to \mathbb{R}^{m}$.

  b) There exists $K > 0$ such that for all $x, y \in D(G)$, $x = col\,(x^a, x^b)$, $y = col\,(y^a, y^b)$ and all $w_1 \in G(x)$, $w_2 \in G(y)$:

$$\langle w_1 - w_2,\, x - y\rangle_{\mathbb{R}^{2n+m}} \geq K \cdot \|x^b - y^b\|^2_{\mathbb{R}^m}.$$

  c) There exists $K_0 > 0$ such that for all x, $y \in D(G)$ and for all $w_1 \in G(x)$, $w_2 \in G(y)$:

$$\langle w_1 - w_2,\, x - y\rangle_{\mathbb{R}^{2n+m}} \geq K_0 \cdot \|x - y\|^2_{\mathbb{R}^{2n+m}}.$$

- (H4) $S = \mathrm{diag}(s_1, s_2, \ldots, s_m)$ with $s_j > 0$, $j = \overline{1,m}$.
- (H5) The functions f, $g \in L^1_{loc}(\mathbb{R}; L^2(0,1))$ and f, g are T_0- periodic in time, that is:

$$f(t + T_0, x) = f(t,x), \quad g(t + T_0, x) = g(t,x), \quad \text{a.a. } (t,x) \in \mathbb{R} \times (0,1).$$

Remark 1. Our assumption (H3)a is a technical one and it is automatically satisfied if G is a matrix.

2. THE CASE $B(t) \equiv b_0$ (A CONSTANT VECTOR)

We can replace G by $\tilde{G}$ defined by $\tilde{G}w = Gw - b_0$, which is also, in the assumption (H3)a, a maximal monotone operator. Therefore, we can assume in this case, without loss of generality, that $B(t) \equiv 0$.

First, we present some existence and uniqueness results for the solutions of the problem (S), (BC), (IC) obtained in [5] (see also [7]). We consider the *spaces* $X = (L^2(0,1))^2$, $\mathbb{R}^m$ and $Y = X \times \mathbb{R}^m$ with the corresponding scalar products:

$$\langle f,g\rangle_X = \int_0^1 f_1 g_1 dx + \int_0^1 f_2 g_2 dx, \ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in X,$$

$$\langle x,y\rangle_s = \sum_{i=1}^{m} s_i x_i y_i, \ x,y \in I\!R^m,$$

$$\langle \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \rangle_Y = \langle f,g\rangle_X + \langle x,y\rangle_s, \ \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \in Y.$$

We define the *operator* $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$,

$$D(\mathcal{A}) = \Bigg\{ u = col(i,v,w) \in Y; \ i,v \in H^n(0,1), \ w \in I\!R^m, \ \begin{pmatrix} \gamma_0 v \\ w \end{pmatrix} \in D(G),$$

$$\gamma_1 i \in -G_{11}(\gamma_0 v) - G_{12}(w) \Bigg\},$$

where $\gamma_0 v = col\,(v(0), v(1), \dots, v^{(n-1)}(0), v^{(n-1)}(1))$, $\gamma_1 i = col\,((L_1 i)(0), -(L_1 i)(1), \dots,$ $(L_n i)(0), -(L_n i)(1))$ and $(L_j i)(x) = \sum_{k=j}^{n} (-1)^{k-j} (a_k i)^{(k-j)}, \ j = \overline{1,n}$,

$$\mathcal{A} \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n} a_k v^{(k)} \\ -\sum_{k=0}^{n} (-1)^k (a_k i)^{(k)} \\ S^{-1} G_{21}(\gamma_0 v) + S^{-1} G_{22}(w) \end{pmatrix}, \ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(\mathcal{A}).$$

If we denote by $Pv = \sum_{k=0}^{n} a_k v^{(k)}$ then the operator $\mathcal{A}$ can be written as:

$$\mathcal{A} \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} Pv \\ -P^* i \\ S^{-1} G_{21}(\gamma_0 v) + S^{-1} G_{22}(w) \end{pmatrix}, \ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(\mathcal{A}),$$

where P^* is the adjoint operator of P.

We also define the *operator* $\mathcal{B} : D(\mathcal{B}) \subset Y \to Y$,

$$\mathcal{B} \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} \alpha(\cdot, i) \\ \beta(\cdot, v) \\ 0 \end{pmatrix}, \ D(\mathcal{B}) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y; \ \mathcal{B} \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y \right\}.$$

Remark 2. Under the assumptions (H1), (H2)a, (H3)a and (H4) we can easily show that $D(\mathcal{A}) \neq \emptyset$, $\overline{D(\mathcal{A})} = X \times \overline{D(G_{12}) \cap D(G_{22})}$ and $D(\mathcal{A}) \subset D(\mathcal{B})$.

Lemma 1. *If* (H1), (H2)a, (H3)a *and* (H4) *hold, then the operator* $\mathcal{A} + \mathcal{B}$ *is maximal monotone.*

For the proof of Lemma 1 see [5] (see also [7]).

Using the operators $\mathcal{A}$ and $\mathcal{B}$ the problem (S), (BC), (IC) can be equivalently expressed as the following Cauchy problem in the space Y:

$$\text{(P)} \qquad \begin{cases} \dfrac{du}{dt}(t) + \mathcal{A}(u(t)) + \mathcal{B}(u(t)) \ni F(t,\cdot), \quad t > 0 \\ u(0) = u_0, \end{cases}$$

where $u = col(i, v, w)$, $F(t,\cdot) = col(f(t,\cdot), g(t,\cdot), 0)$, $u_0 = col(i_0, v_o, w_0)$, $w_0 = col(w_{10}, \dots, w_{m0})$.

We say that $col(i, v, w)$ is a strong (weak) solution for the problem (S), (BC) if $col(i, v, w)$ is a strong (respectively weak) solution for the equation $(\mathrm{P})_1$, (see [1,Ch.III]).

Theorem 1 *a) Assume that (H1), (H2)a, (H3)a and (H4) hold. If* i_0, $v_0 \in H^n(0,1)$, $w_0 \in \mathbb{R}^m$, $\begin{pmatrix} \gamma_0 v_0 \\ w_0 \end{pmatrix} \in D(G)$, $\gamma_1 i_0 \in -G_{11}(\gamma_0 v_0) - G_{12}(w_0)$ *and* f, $g \in W^{1,1}(0,T; L^2(0,1))$ *(with* $T > 0$ *fixed), then the problem (S), (BC), (IC) has a unique strong solution* $u = col(i, v, w) \in W^{1,\infty}(0,T; Y)$. *Moreover:*

$$i,\ v \in L^\infty(0,T; H^n(0,1)), \ \ hence\ \ \partial^j i/\partial x^j,\ \ \partial^j v/\partial x^j \in L^\infty((0,T) \times (0,1)), \quad j = \overline{0, n-1}. \tag{1}$$

b) Assume that (H1), (H2)a, (H3)a and (H4) hold. If $i_0, v_0 \in L^2(0,1)$, $w_0 \in \overline{D(G_{12}) \cap D(G_{22})}$ *and* f, $g \in L^1(0,T; L^2(0,1))$, *(with* $T > 0$ *fixed), then the problem (S), (BC), (IC) has a unique weak solution* $u = col(i, v, w) \in C([0,T]; Y)$.

Proof. Using Lemma 1, the operator $\mathcal{A} + \mathcal{B} : D(\mathcal{A}) \subset Y \to Y$ is maximal monotone. Then, by the general theory of evolution equations of monotone type in Hilbert spaces, we deduce the conclusions a) and b) of the theorem. It remains to prove (1). For, let us consider the equation $(\mathrm{P})_1$ in the interval $[0, T+\varepsilon]$, $\varepsilon > 0$ (by extending correspondingly the functions f and g). Then $u(T) = col(i(T), v(T), w(T)) \in D(\mathcal{A})$ and we have:

$$\|(\mathcal{A} + \mathcal{B})(u(t))\|_Y \le \|F(t,\cdot)\|_Y + \left\| \frac{d^+ u}{dt}(t) \right\|_Y, \quad 0 \le t \le T.$$

By the above relation and by the fact that $du/dt \in L^\infty(0,T; Y)$, we deduce that:

$$\sup\{\|(\mathcal{A} + \mathcal{B})(u(t))\|_Y;\ 0 \le t \le T\} \le const. \tag{2}$$

($const.$ is a positive constant).

By (2) and [10, Lemma B], using a similar argument as that used in the proof of Lemma 1 (see [5]), we obtain that $\partial^j i/\partial x^j$, $\partial^j v/\partial x^j \in L^\infty(0,T; C([0,1]))$, $j = \overline{0, n-1}$ and $\partial^n i/\partial x^n$, $\partial^n v/\partial x^n \in L^\infty(0,T; L^1(0,1))$. By (H2)a we deduce that $\alpha(\cdot, i)$, $\beta(\cdot, v) \in L^\infty(0,T; L^2(0,1))$ and by the equations of (S) we obtain $\partial^n i/\partial x^n$, $\partial^n v/\partial x^n \in L^\infty(0,T; L^2(0,1))$. Hence i, $v \in L^\infty(0,T; H^n(0,1))$ and by the Sobolev's imbedding theorem we get $\partial^j i/\partial x^j$, $\partial^j v/\partial x^j \in L^\infty((0,T) \times (0,1))$, $j = \overline{0, n-1}$. Q.E.D.

Lemma 2. *Assume that* (H1), (H2)ab, (H3)ab *and* (H4) *hold. Then the operator* $\mathcal{A} + \mathcal{B}$ *is coercive with respect to any* $u_0 = col(i_0, v_0, w_0) \in D(\mathcal{A})$, *that is:*

$$\lim_{\substack{\|u\|_Y \to \infty \\ u \in D(\mathcal{A})}} \frac{\langle(\mathcal{A} + \mathcal{B})(u), u - u_0\rangle_Y}{\|u\|_Y} = \infty. \tag{3}$$

Proof. We suppose without loss of generality that G is single-valued. Let $u_0 = col(i_0, v_0, w_0)$ be arbitrary, but fixed in $D(\mathcal{A})$. Using the assumption (H3)b, for every $u = col(i,v,w) \in D(\mathcal{A})$, we have:

$$\langle(\mathcal{A}+\mathcal{B})(u), u-u_0\rangle_Y = \langle\mathcal{A}(u)-\mathcal{A}(u_0), u-u_0\rangle_Y + \langle\mathcal{B}(u), u-u_0\rangle_Y + \underbrace{\langle\mathcal{A}(u_0), u-u_0\rangle_Y}_{E_0} =$$

$$= \langle G\begin{pmatrix}\gamma_0 v\\ w\end{pmatrix} - G\begin{pmatrix}\gamma_0 v_0\\ w_0\end{pmatrix}, \begin{pmatrix}\gamma_0(v-v_0)\\ w-w_0\end{pmatrix}\rangle_{\mathbb{R}^{2n+m}} + \int_0^1 \alpha(x,i(x))\cdot(i(x)-i_0(x))\,dx +$$
$$+\int_0^1 \beta(x,v(x))\cdot(v(x)-v_0(x))\,dx + E_0 \geq K\cdot\|w-w_0\|^2_{\mathbb{R}^m} + \int_0^1 \alpha(x,i(x))\cdot(i(x)-$$
$$-i_0(x))\,dx + \int_0^1 \beta(x,v(x))\cdot(v(x)-v_0(x))\,dx + E_0.$$

Therefore, for $u \neq 0$, we obtain:

$$\frac{\langle(\mathcal{A}+\mathcal{B})u, u-u_0\rangle_Y}{\|u\|_Y} \geq \frac{K\cdot\|w-w_0\|^2_{\mathbb{R}^m}}{\|i\|_{L^2(0,1)}+\|v\|_{L^2(0,1)}+\|w\|_s} + \frac{E_0}{\|u\|_Y} +$$
$$+\frac{\displaystyle\int_0^1 \alpha(x,i(x))\cdot(i(x)-i_0(x))\,dx + \int_0^1 \beta(x,v(x))\cdot(v(x)-v_0(x))\,dx}{\|i\|_{L^2(0,1)}+\|v\|_{L^2(0,1)}+\|w\|_s} \geq$$
$$\geq \min\left\{\frac{\displaystyle\int_0^1 \alpha(x,i(x))\cdot(i(x)-i_0(x))\,dx}{\|i\|_{L^2(0,1)}}, \frac{\displaystyle\int_0^1 \beta(x,v(x))\cdot(v(x)-v_0(x))\,dx}{\|v\|_{L^2(0,1)}},\right.$$
$$\left.\frac{K\|w-w_0\|^2_{\mathbb{R}^m}}{\|w\|_s}\right\} + \frac{E_0}{\|u\|_Y}.$$

To prove (3) it is sufficient to show that:

$$\lim_{\|i\|_{L^2(0,1)}\to\infty} \frac{\displaystyle\int_0^1 \alpha(x,i(x))\cdot(i(x)-i_0(x))\,dx}{\|i\|_{L^2(0,1)}} = \infty, \tag{4}$$

$$\lim_{\|v\|_{L^2(0,1)}\to\infty} \frac{\displaystyle\int_0^1 \beta(x,v(x))\cdot(v(x)-v_0(x))\,dx}{\|v\|_{L^2(0,1)}} = \infty \tag{5}$$

and

$$\lim_{\|w\|_s\to\infty} \frac{K\cdot\|w-w_0\|^2_{\mathbb{R}^m}}{\|w\|_s} = \infty. \tag{6}$$

For the relation (4), using the assumptions (H2)ab we have:

$$\alpha(x,i(x))\cdot(i(x)-i_0(x)) \geq |\alpha(x,i(x))-\alpha(x,i_0(x))|\cdot|i(x)-i_0(x)|-$$
$$-|\alpha(x,i_0(x))|\cdot|i(x)-i_0(x)| \geq (|\alpha(x,i(x))|-|\alpha(x,i_0(x))|)\cdot|i(x)-i_0(x)|-$$
$$-|\alpha(x,i_0(x))|\cdot|i(x)-i_0(x)| \geq (a|i(x)|-c(x))\cdot|i(x)-i_0(x)|-2|\alpha(x,i_0(x))|\cdot(|i(x)|+$$
$$+|i_0(x)|) \geq a|i(x)|\cdot(|i(x)|-|i_0(x)|)-c(x)\cdot(|i(x)|+|i_0(x)|)-2|\alpha(x,i_0(x))|\cdot(|i(x)|+$$
$$+|i_0(x)|) \geq ai^2(x)-C_1i^2(x)-C_2(a|i_0(x)|+c(x)+2|\alpha(x,i_0(x))|)^2-c^2(x)-i_0^2(x)-$$
$$-4\alpha^2(x,i_0(x))-i_0^2(x) = \tilde{a}i^2(x)-C_3(i_0^2(x)+c^2(x)+\alpha^2(x,i_0(x))),\ 0<x<1,$$

(we choose C_1, $C_2 > 0$ such that $C_1 < a$; $\tilde{a} = a - C_1 > 0$, $C_3 > 0$).

We integrate the above inequality over $[0,1]$ and we get:

$$\int_0^1 \alpha(x, i(x)) \cdot (i(x) - i_0(x))\, dx \geq \tilde{a} \int_0^1 i^2(x)\, dx - C_3 \int_0^1 (i_0^2(x) + c^2(x) + \\ + \alpha^2(x, i_0(x)))\, dx = \tilde{a}\|i\|^2_{L^2(0,1)} - C_4, \quad C_4 > 0,$$

$(i_0, c, \alpha(\cdot, i_0) \in L^2(0,1))$.

From the obtained inequality we deduce (4). In the same manner we deduce the relation (5). The last relation (6) is a simple consequence of the equivalence between the norms $\|\cdot\|_{\mathbb{R}^m}$ and $\|\cdot\|_s$. Q.E.D.

Theorem 2. *Assume that* (H1), (H2)ab, (H3)ab, (H4) *and* (H5) *hold. Then the problem* (S), (BC) *has at least one* T_0*-periodic weak solution.*

Proof. Let $u_0 = col(i_0, v_0, w_0) \in D(\mathcal{A})$ be fixed. We define the operator $\widetilde{\mathcal{A}+\mathcal{B}}$ by:

$$D(\widetilde{\mathcal{A}+\mathcal{B}}) = \{u \in Y,\ u + u_0 \in D(\mathcal{A}+\mathcal{B})\}, \quad (\widetilde{\mathcal{A}+\mathcal{B}})(u) = (\mathcal{A}+\mathcal{B})(u + u_0).$$

By Lemma 2 the operator $\widetilde{\mathcal{A}+\mathcal{B}}$ is coercive with respect to $0 \in Y$. We make a change of functions: $\tilde{i}(t,x) = i(t,x) - i_0(x)$, $\tilde{v}(t,x) = v(t,x) - v_0(x)$, $\tilde{w}(t) = w(t) - w_0$. Then the problem (S), (BC) becomes:

$$\text{(E)} \qquad \frac{d\tilde{u}}{dt}(t) + (\widetilde{\mathcal{A}+\mathcal{B}})(\tilde{u}(t)) \ni F(t, \cdot).$$

Using the assumption (H5) and [3, Corollary 1, p.207] we deduce that the solutions of the equation (E) are bounded on the positive half-axis. Hence, all the solutions of the equation $(\mathrm{P})_1$ are also bounded.

Now, we define the operator $\mathcal{L} : \overline{D(\mathcal{A})} \to \overline{D(\mathcal{A})}$, $\mathcal{L}(u_0) = u(T_0; u_0)$, where $u(t; u_0)$, $t \geq 0$ is the weak solution of the problem (S), (BC) with the initial date u_0. The operator $\mathcal{L}$ is nonexpansive and for $u_0 \in \overline{D(\mathcal{A})}$, the sequence $\{\mathcal{L}^n(u_0)\}_{n \geq 1}$ is bounded in Y. By F. Browder and W.V. Petryshyn's theorem (see [2]) we deduce that the operator $\mathcal{L}$ has at least one fixed point. Therefore the problem (S), (BC) has at least one time periodic weak solution with the period T_0. Q.E.D.

Remark 3. If $\alpha(x, \cdot)$ and $\beta(x, \cdot)$ are strongly monotone, that is the operator $\mathcal{A} + \mathcal{B}$ is strongly monotone, then, under the assumptions of the Theorem 2, the problem (S), (BC) has a unique T_0-periodic strong solution, as soon as f, $g \in W^{1,1}(0, T_0; L^2(0,1))$ are T_0-periodic functions in t.

3. THE CASE $B(t) \not\equiv const.$

We suppose here, without loss of generality that $f = g \equiv 0$. Then, we shall study the existence of periodic weak solutions, under the assumption that b_k are periodic functions. Some existence, uniqueness and regularity properties of the solutions of the problem (S), (BC), (IC) were obtained in [5] by the change of function $i = \tilde{i} + \breve{i}$, with $\breve{i}(t,x) = A_1(t)x^{2n-1} + A_2(t)x^{2n-2} + \cdots + A_{2n-1}(t)x + A_{2n}(t)$, where $A_k(t)$, $k = \overline{1, 2n}$, are uniquely determined by the system: $\gamma_1 \breve{i} = B_1(t)$, $(B(t) = col(B_1(t), B_2(t))$, $B_1(t) = col(b_1(t), \ldots, b_{2n}(t))$, $B_2(t) = col(b_{2n+1}(t), \ldots, b_{2n+m}(t)))$.

Then the problem (S), (BC), (IC) can be written as:

$$(\tilde{\mathrm{S}}) \qquad \begin{cases} \dfrac{\partial \tilde i}{\partial t} + \displaystyle\sum_{k=0}^{n} a_k(x)\frac{\partial^k v}{\partial x^k} + \alpha(x, \tilde i + \bar i) = \tilde f(t,x) \\ \dfrac{\partial v}{\partial t} - \displaystyle\sum_{k=0}^{n} (-1)^k \frac{\partial^k}{\partial x^k}[a_k(x)\tilde i] + \beta(x,v) = \tilde g(t,x), \\ \qquad\qquad 0 < x < 1,\ t > 0, \end{cases}$$

with the boundary condition:

$(\widetilde{\mathrm{BC}})$

$$\begin{pmatrix} col((L_1\tilde i)(t,0), -(L_1\tilde i)(t,1), \dots, (L_n\tilde i)(t,0), -(L_n\tilde i)(t,1)) \\ col\,(s_1 w_1'(t), \dots, s_m w_m'(t)) \end{pmatrix} \in$$

$$\in -G\begin{pmatrix} col(v(t,0), v(t,1), \dots, \dfrac{\partial^{n-1}v}{\partial x^{n-1}}(t,0), \dfrac{\partial^{n-1}v}{\partial x^{n-1}}(t,1)) \\ col\,(w_1(t), \dots, w_m(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ B_2(t) \end{pmatrix}, \quad t > 0$$

and the initial data:

$$(\widetilde{\mathrm{IC}}) \qquad \begin{cases} \tilde i(0,x) = \tilde i_0(x),\ v(0,x) = v_0(x),\ 0 < x < 1, \\ w_j(0) = w_{j0},\ j = \overline{1,m}, \end{cases}$$

where $\tilde f(t,x) = -\dfrac{\partial \bar i}{\partial t}(t,x)$, $\tilde g(t,x) = \displaystyle\sum_{k=0}^{n}(-1)^k \frac{\partial^k}{\partial x^k}[a_k(x)\bar i]$, $0 < x < 1$, $t > 0$, $\tilde i_0(x) = i_0(x) - \bar i(0,x)$, $0 < x < 1$.

Using once again the operators $\mathcal{A}$ and $\mathcal{B}$, the problem $(\tilde{\mathrm{S}})$, $(\widetilde{\mathrm{BC}})$, $(\widetilde{\mathrm{IC}})$ leads us to consider in the space Y the following time dependent Cauchy problem, equivalent to (P):

$$(\tilde{\mathrm{P}}) \qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} \tilde i \\ v \\ w \end{pmatrix} + \mathcal{A}\begin{pmatrix} \tilde i \\ v \\ w \end{pmatrix} + \mathcal{B}\begin{pmatrix} \tilde i + \bar i(t) \\ v \\ w \end{pmatrix} \ni \begin{pmatrix} \tilde f(t,\cdot) \\ \tilde g(t,\cdot) \\ S^{-1}B_2(t) \end{pmatrix}, \quad t > 0 \\ \begin{pmatrix} \tilde i(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} \tilde i_0 \\ v_0 \\ w_0 \end{pmatrix} \end{cases}$$

Theorem 3. a) *Assume that* (H1), (H2)a, (H3)ac *and* (H4) *hold. If* $b_k \in W^{1,2}(0,T)$, $k = \overline{1, 2n+m}$ ($T > 0$ *fixed*), $i_0, v_0 \in H^n(0,1)$, $w_0 \in \mathbb{R}^m$, $\begin{pmatrix} \gamma_0 v_0 \\ w_0 \end{pmatrix} \in D(G)$ *and* $B_1(0) \in \gamma_1 i_0 + G_{11}(\gamma_0 v_0) + G_{12}(w_0)$, *then the problem* (S), (BC), (IC) *has a unique strong solution* $u = col(i,v,w) \in W^{1,\infty}(0,T;Y)$. *Moreover* $i, v \in L^\infty(0,T;H^n(0,1))$.

b) *Assume that* (H1), (H2)a, (H3)ac *and* (H4) *hold. If* $b_k \in L^2(0,T)$, $k = \overline{1, 2n+m}$ ($T > 0$ *fixed*), $i_0, v_0 \in L^2(0,1)$, $w_0 \in D(G_{12}) \cap D(G_{22})$, *then the problem* (S), (BC), (IC) *has a unique weak solution* $u = col(i,v,w) \in C([0,T];Y)$.

Proof. We suppose again that G is single-valued. As in [8, Th.1] we assume that $b_k \in W^{2,\infty}(0,T)$, $k = \overline{1,2n}$, $b_k \in W^{1,\infty}(0,T)$, $k = \overline{2n+1, 2n+m}$ and $\alpha(x,\cdot)$ is Lipschitz continuous with Lipschitz constant independent on x. Using Kato's theorem (see [4]) it follows that the problem $(\widetilde{\mathrm{P}})$ has a unique strong solution $\tilde{u} = col(\tilde{\imath}, v, w) \in W^{1,\infty}(0,T;Y)$. Moreover $\tilde{u}$ is differentiable from the right on $[0,T)$ and the function $u = col(i,v,w)$ satisfies:

$$\begin{cases} \dfrac{d^+u}{dt}(t) + \mathcal{A}(u(t)) + \mathcal{B}(u(t)) = col(0,0,S^{-1}B_2(t)), \\ (\gamma_1 i)(t) = -G_{11}((\gamma_0 v)(t)) - G_{12}(w(t)) + B_1(t), \quad 0 \le t < T \\ u(0) = u_0, \end{cases} \tag{7}$$

that is u is a solution of the problem (S), (BC), (IC).

Now, we suppose that α is not a Lipschitz continuous function and we replace α by the Yosida approximate $\alpha_\lambda(x,\cdot)$, $\lambda > 0$. From the above reasoning we deduce that the problem $(\widetilde{\mathrm{P}})$ with α_λ instead of α has a unique strong solution $u_\lambda = col(i_\lambda, v_\lambda, w_\lambda) \in W^{1,\infty}(0,T;Y)$. Then u_λ verifies the following problem:

$$\begin{cases} \dfrac{d^+u_\lambda}{dt}(t) + \mathcal{A}(u_\lambda(t)) + \mathcal{B}_\lambda(u_\lambda(t)) = col(0,0,S^{-1}B_2(t)) \\ (\gamma_1 i_\lambda)(t) = -G_{11}((\gamma_0 v_\lambda)(t)) - G_{12}(w_\lambda(t)) + B_1(t), \quad 0 \le t < T \\ u_\lambda(0) = u_0 \end{cases} \tag{8}$$

with $\mathcal{B}_\lambda(u) = col(\alpha_\lambda(\cdot,i),\beta(\cdot,v),0)$, $\lambda > 0$.

Using (8) and the assumptions of the theorem, after some computations we deduce that:

$$\left\|\frac{du_\lambda}{dt}(t)\right\|_Y \le const., \quad \|u_\lambda(t)\|_Y \le const., \ 0 < t < T, \ \lambda > 0 \quad \text{and}$$
$$\int_0^1 [|(Pv_\lambda)(t,x)| + |(P^* i_\lambda)(t,x)|]\, dx \le const., \ t > 0, \ \lambda > 0.$$

By [10, Lemma B] we obtain:

$$\partial^j i_\lambda/\partial x^j, \ \partial^j v_\lambda/\partial x^j \ \text{ are bounded in } L^\infty((0,T)\times(0,1)), \ j = \overline{0,n-1}$$
$$\partial^n i_\lambda/\partial x^n, \ \partial^n v_\lambda/\partial x^n \ \text{ are bounded in } L^\infty(0,T;L^1(0,1)) \ \text{ and}$$
$$\mathcal{B}_\lambda(u_\lambda) \ \text{ is bounded in } L^\infty(0,T;Y).$$

Next, by (8) we have:

$$\|u_\lambda(t) - u_\mu(t)\| \le const.(\lambda+\mu)^{1/2}, \ 0 \le t \le T, \ \lambda, \mu > 0,$$

that is the sequence (u_λ) converges to some $u = col(i,v,w)$ in $C([0,T];Y)$, as $\lambda \to 0$.

By Lebesgue's Dominated Convergence Theorem we can prove that:

$$\mathcal{B}_\lambda(u_\lambda) \to \mathcal{B}(u), \ \text{ as } \lambda \to 0, \ \text{ strongly in } L^2(0,T;Y).$$

Since $\mathcal{A}$ and G are closed, by letting $\lambda \to 0$ in (8), we conclude that u is a strong solution of the problem (S), (BC), (IC).

For the general case $b_k \in W^{1,2}(0,T)$, $k = \overline{1,2n+m}$, the regularity properties of the solution and for the part b) of the theorem we use standard arguments (see [5,8]). Q.E.D.

We give now some sufficient conditions for the boundedness of the solutions of the problem (S), (BC):

Theorem 4. *Assume that* (H1), (H2)ab, (H3)ac, (H4) *hold and* $b_k \in L^\infty(\mathbb{R}_+)$, $k = \overline{1,2n+m}$. *Then every weak solution of the problem* (S), (BC) *is bounded on* $\mathbb{R}_+$.

Proof. By Lemma 1 and Lemma 2 the operator $\mathcal{A} + \mathcal{B}$ is maximal monotone and coercive and then $R(\mathcal{A}+\mathcal{B}) = Y$, hence $F = (\mathcal{A}+\mathcal{B})^{-1}(0) \neq \emptyset$. We suppose that G is single-valued.

We use a similar idea as that we used in [6] (see also [5]). First, we shall show that if $b_k \in W^{1,2}_{loc}(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ then the strong solution of the problem (S), (BC) is bounded on $\mathbb{R}_+$. To prove this, let us consider $T > 0$ arbitrary, but fixed for the moment, $b_k \in W^{1,2}(0,T) \cap L^\infty(\mathbb{R}_+)$, $k = \overline{1, 2n+m}$, i_0, $v_0 \in H^n(0,1)$, $w_0 \in \mathbb{R}^m$, $\begin{pmatrix} \gamma_0 v_0 \\ w_0 \end{pmatrix} \in D(G)$ and $B_1(0) \in \gamma_1 i_0 + G_{11}(\gamma_0 v_0) + G_{12}(w_0)$. Therefore, the corresponding strong solution $u(t) = col(i(t), v(t), w(t))$ of the problem (S), (BC), (IC) satisfies (7).

Using an element $\gamma = col(p,q,r) \in F$, that is $(\mathcal{A}+\mathcal{B})(\gamma) = 0$, by (7) we have:

$$\frac{1}{2}\frac{d}{dt}\|u(t)-\gamma\|^2_Y + \langle G\begin{pmatrix} (\gamma_0 v)(t) \\ w(t) \end{pmatrix} - G\begin{pmatrix} \gamma_0 q \\ r \end{pmatrix}, \begin{pmatrix} \gamma_0(v(t)-q) \\ w(t)-r \end{pmatrix}\rangle_{\mathbb{R}^{2n+m}} +$$
$$+\int_0^1 [\alpha(x,i(t,x)) - \alpha(x,p(x))]\cdot[i(t,x)-p(x)]\,dx + \int_0^1 [\beta(x,v(t,x)) - \beta(x,q(x))]\cdot$$
$$\cdot[v(t,x)-q(x)]\,dx = \langle B_1(t), (\gamma_0 v)(t) - \gamma_0 q\rangle_{\mathbb{R}^{2n}} + \langle B_2(t), w(t)-r\rangle_{\mathbb{R}^m},\ \ 0 \le t < T.$$

By the assumption (H3)c, the above relation give us:

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|u(t)-\gamma\|^2_Y + C_5\|w(t)-r\|^2_{\mathbb{R}^m} + \int_0^1 [\alpha(x,i(t,x)) - \alpha(x,p(x))]\cdot[i(t,x)- \\ -p(x)]\,dx + \int_0^1 [\beta(x,v(t,x)) - \beta(x,q(x))]\cdot[v(t,x)-q(x)]\,dx \le \\ \le C_6\|B_2(t)\|^2_{\mathbb{R}^{2n+m}} \le C_7,\ \ 0 \le t < T,\end{gathered} \tag{9}$$

where C_5, C_6, $C_7 > 0$ are independent of T.

By the assumptions (H2)ab, we have:

$$[\alpha(x,i(t,x)) - \alpha(x,p(x))]\cdot[i(t,x)-p(x)] \ge [a|i(t,x)| - c(x)]\cdot|i(t,x)-p(x)| -$$
$$-|\alpha(x,p(x))|\cdot|i(t,x)-p(x)| \ge a|i(t,x)-p(x)|^2 - (a|p(x)| + c(x) + |\alpha(x,p(x))|)\cdot$$
$$\cdot|i(t,x)-p(x)| \ge \tilde{a}|i(t,x)-p(x)|^2 - C_8(a^2p^2(x) + c^2(x) + |\alpha(x,p(x))|^2),$$

$0 < x < 1$, $0 \le t \le T$, (where $\tilde{a}$, $C_8 > 0$ are independent of T).

We integrate the above inequality over $[0,1]$ and we obtain:

$$\begin{gathered}\int_0^1 [\alpha(x,i(t,x)) - \alpha(x,p(x))]\cdot[i(t,x)-p(x)]dx \ge \\ \ge \tilde{a}\,\|\,i(t)-p\,\|^2_{L^2(0,1)} - C_9,\ \ 0 \le t \le T.\end{gathered} \tag{10}$$

In the same manner, we have:

$$\begin{gathered}\int_0^1 [\beta(x,v(t,x)) - \beta(x,q(x))]\cdot[v(t,x)-q(x)]dx \ge \\ \ge \tilde{b}\,\|\,v(t)-q\,\|^2_{L^2(0,1)} - C_{10},\ \ 0 \le t \le T,\end{gathered} \tag{11}$$

with $\tilde{b}$, C_9, $C_{10} > 0$ independent of T.

From the inequalities (9), (10) and (11) we deduce:

$$\frac{1}{2}\frac{d}{dt}\|u(t)-\gamma\|^2_Y + C_5\,\|\,w(t)-r\,\|^2_{\mathbb{R}^m} + \tilde{a}\,\|\,i(t)-p\,\|^2_{L^2(0,1)} +$$
$$+\tilde{b}\,\|\,v(t)-q\,\|^2_{L^2(0,1)} \le C_7 + C_9 + C_{10},\ \ 0 \le t < T.$$

So:

$$\frac{1}{2}\frac{d}{dt}\|u(t)-\gamma\|_Y^2 + C_{11}\|u(t)-\gamma\|_Y^2 \le C_{12},\ \ 0 \le t < T,$$

where the constants $C_{11}, C_{12} > 0$ are independent of T.

Integrating over $[0,t]$ the above inequality, we get:

$$\|u(t)-\gamma\|_Y^2 \le e^{-2C_{11}t}\,\|u_0-\gamma\|_Y^2 + \frac{C_{12}}{C_{11}}\left(1-e^{-2C_{11}t}\right),\ \ 0 \le t \le T. \tag{12}$$

Therefore:

$$\|u(t)-\gamma\|_Y \le \text{const.},\ \ 0 \le t \le T,$$

where const. is a positive constant independent of T. Because T is arbitrary we conclude that the solution $col(i,v,w)$ is bounded on $\mathbb{R}_+$.

Because the inequality (12) remains also true for the weak solutions, we deduce the conclusion of the theorem. Q.E.D.

Theorem 5. *Assume that (H1), (H2)a, (H3)ac, (H4) hold and* $b_k \in L^2_{loc}(\mathbb{R}_+)$, $k = \overline{1, 2n+m}$ *are* T_0*-periodic functions. Then, if the problem (S), (BC) has at least one bounded solution on* $\mathbb{R}_+$, *then the problem has also a weak* T_0*-periodic solution.*

Proof. Let $u = col(i,v,w)$ a bounded solution of the problem (S), (BC). Then, by the inequality:

$$\|u(t)-\bar{u}(t)\|_Y \le \|u_0-\bar{u}_0\|_Y,\ \ t > 0$$

we deduce that all the solutions of (S), (BC) are bounded on $\mathbb{R}_+$. Using now the operator $\mathcal{L}$, defined as in the proof of Theorem 2 (for this case) and the same fixed point theorem we conclude that the problem (S), (BC) has at least one T_0-periodic weak solution. Q.E.D.

By Theorem 4 and Theorem 5 we deduce:

Corollary. *Assume that (H1), (H2)ab, (H3)ac, (H4) hold and* $b_k \in L^\infty_{loc}(\mathbb{R}_+)$, $k = \overline{1, 2n+m}$, T_0*-periodic functions. Then the problem (S), (BC) has at least a time periodic weak solution with period* T_0.

Remark 4. If $\alpha(x,p) = C_1 p|p|^{a-1}$, $\beta(x,p) = C_2 p|p|^{b-1}$, $a,b \ge 1$, $C_1, C_2 > 0$, then an easy verification shows that α and β satisfy the conditions (H2)ab. For other examples, see [11].

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