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POTENTIALS ON NONORIENTABLE KLEIN SURFACES

Dorin Ghisa

To Professor Corneliu Constantinescu

Abstract

Some concepts related to the potential theory on Riemann surfaces are extended to nonorientable Klein surfaces. A relationship between Kerékjártó-Stoilow compactification of a nonorientable Klein surface and that of its orientable double cover is established.

1. Introduction

Riemann surfaces will mean in this paper both *bordered and border free* Riemann surfaces. A Riemann surface R endowed with a *fixed-point free* antianalytic involution h will be called *symmetric Riemann surface* and h will be called *a symmetry.* The notation (R, h) for such a surface will be used.

It is known (see [2]) that the quotient space $S = R/\langle h\rangle$, where $\langle h\rangle$ is the two element group generated by h, carries a unique *dianalytic structure* which makes the canonical projection *a dianalytic function* and the space S a *nonorientable Klein surface.* If h' and h'' are different symmetries of R, then $R/\langle h'\rangle$ and $R/\langle h''\rangle$ are dianalytically equivalent and they will be considered identical. Vice-versa, to every nonorientable Klein surface S, we can associate a symmetric Riemann surface (R, h) such that $R/\langle h\rangle$ and S are dianalytically equivalent. R is called *the orientable double cover* of S. If $\pi : R \to S$ is the canonical projection, then every point $\tilde{a} \in S$ is the image of exactly two points a and $h(a)$ of R.

There is an obvious isomorphism given by $f = \tilde{f}\circ\pi$ between functions $\tilde{f}$ on S and h-invariant functions f on R. Isomorphisms have also been established between metrics on S and h-invariant metrics of R (see [5]) and between vector fields on S and h-invariant vector fields on R (see [3]). In this paper we will define h-invariant Green's functions, which are in fact lifts on R of ordinary Green's functions defined on S. We will also study the relationship between R and S in terms of those Green's functions, of Borel measures, potentials and compactifications. The main bibliographic source is the monograph [6] by C. Constantinescu and A. Cornea.

2. h-Invariant Green's Functions

If R is a hyperbolic Riemann surface and $a \in R$, let g_a be the Green's function with the pole in a. It is known (see [6]) that g_a is characterized by the following properties:

(a) g_a is harmonic up to the point a, where it has a logarithmic singularity with coefficient 1.

(b) If $G \subsetneq R$ is a domain and $a \in G$, then $g_a = H_{g_a}^G$ in G, where $H_{g_a}^G$ represents the solution of the Dirichlet problem with boundary value g_a on ∂G.

For a symmetric Riemann surface (R, h), let us consider the function

(1) $$b \rightarrow g_a(b) + g_{h(a)}(b)$$

Given the uniqueness of Green's function with a prescribed pole, we have the equality:

$$g_a(b) = g_{h(a)}(h(b)), \text{ and therefore:}$$

$$g_a(b) = g_{h(a)}(b) = g_{h(a)}(h(b)) + g_a(h(b)).$$

This shows that the function (1) is h-invariant. We use the notation $g_{\langle a \rangle}$ for this function and call it the h-invariant Green's function with the poles in a and $h(a)$. For the case where R is the annulus an explicit formula for $g_{\langle a \rangle}$ has been found in [4].

Theorem 1 *The h-invariant Green's function fulfills the following symmetry property:*

$$g_{\langle a \rangle}(b) = g_{\langle b \rangle}(a), \qquad \textit{for every } a \textit{ and } b \textit{ in } R.$$

Proof: Taking into account the symmetry property of the ordinary Green's function, we can write:

$$\begin{aligned} g_{\langle a \rangle}(b) &= g_a(b) + g_{h(a)}(b) = g_b(a) + g_b(h(a)) \\ &= g_b(a) + g_{h(b)}(a) = g_{\langle b \rangle}(a). \end{aligned}$$

Let $a \in R$ and let (V, φ) be a parametric disk (or half-disk, if a is on the border of R) centered at a, with V small enough for π to be injective on it. Then $(h(V), \varphi \circ h)$ is a parametric disk (respectively half disk) on R centered at $h(a)$ and $(\pi(V), \varphi \circ \pi_{|V}^{-1})$ is a parametric disk (respectively half-disk) on S centered at $\pi(a)$.

It is known (see [2]) that π is a dianalytic function and therefore if f is an h-invariant harmonic (subharmonic, superharmonic) function in V, then $\tilde{f} = f \circ \pi_{|V}^{-1}$ is a harmonic (respectively subharmonic, superharmonic) function in $\pi(V)$. Vice-versa, if $\tilde{f}$ is harmonic (subharmonic, superharmonic) in $\pi(V)$, then $f = \tilde{f} \circ \pi$ is an h-invariant harmonic (subharmonic, superharmonic) function in $V \cup h(V)$. In particular, a nonorientable Klein surface is hyperbolic if and only if its orientable double cover is hyperbolic. Moreover, the canonical projection $\pi : R \rightarrow S$ induces a one-to-one correspondence between the hyperbolic domains of the nonorientable (not necessarily hyperbolic) Klein surface S and the symmetric hyperbolic domains of its orientable double cover R. The use of Green's functions on S can be reduced to the use of h-invariant Green's functions on R. Indeed, since $g_{\langle a \rangle}$ is an h-invariant harmonic (up to $\{a, h(a)\}$) function on R, there is a unique function u on S such that $g_{\langle a \rangle} = u \circ \pi$ and u is harmonic in S up to $\tilde{a} = \pi(a) = \pi(h(a))$, where it has a logarithmic singularity with coefficient 1. Then $u = g_{\tilde{a}}$ is the Green's function on S with the pole in $\tilde{a}$, in other words $g_{\langle a \rangle} = g_{\tilde{a}} \circ \pi$.

3. h-Invariant Borel Measures

If $A \subseteq R$ is a Borel set, then $h(A)$ is also a Borel set and $A \cup h(A)$ is a symmetric Borel set, which projects by π into a Borel set on S. Vive-versa, for every Borel set $B \subseteq S$, the set $\pi^{-1}(B)$, is a symmetric Borel set in R. The one-to-one correspondence between symmetric Borel subsets of R and Borel subsets of S allows us to establish a one-to-one correspondence between h-invariant Borel measures μ on R and Borel measures $\tilde{\mu}$ on S.

Measures on Riemann surfaces generate potentials by the intermediate of Green's function. Potential theory on Riemann surfaces is a well-studied field, while little is known about potentials on nonorientable Klein surfaces. Obviously, a potential theory on Klein surfaces could be built from scratch, but it is more economical to use the knowledge about potentials on Riemann surfaces and the fact that Klein surfaces are obtained from Riemann surfaces by a factorization. Moreover, most of the theorems of local character should be the same on the two types of surfaces.

Suppose that $\tilde{f}$ is a function of class $C_0^2(S)$. Then $f = \tilde{f} \circ \pi$ is an h-invariant function of class $C_0^2(R)$, and (see [6]) for every $a \in R$

$$-\frac{1}{2\pi}\int g_a d^* df = f(a) = \tilde{f}(\tilde{a}) = f(h(a)) = -\frac{1}{2\pi}\int g_{h(a)} d^* df, \text{ therefore}$$

$$\begin{aligned} \tilde{f}(\tilde{a}) &= \frac{1}{2}\left[-\frac{1}{2\pi}\int g_a d^* df - \frac{1}{2\pi}\int g_{h(a)} d^* df\right] \\ &= -\frac{1}{2\pi}\int g_{(a)} d^* df = -\frac{1}{2\pi}\int g_{\tilde{a}} d^* d\tilde{f}. \end{aligned} \tag{2}$$

It follows that the positive part of $-\frac{1}{2\pi} d^* d\tilde{f}$ and of $\frac{1}{2\pi} d^* d\tilde{f}$ appear as (positive) measures on S and f is the difference of the potentials generated by these measures.

The three theorems below reveal the isomorphism between the family of Borel measures on S and that of h-invariant Borel measures on R.

Theorem 2 *Suppose that an arbitrary Borel measure $\tilde{\mu}$ is given on S. Then, there is a unique h-invariant Borel measure μ on R such that for every Borel set $A \subseteq R$ on which π is injective, $\mu(A) = \tilde{\mu}(\pi(A))$.*

Proof: Since π a local homeomorphism and R has a countable basis, there is a countable family of open sets V_n such that π is injective on every V_n and $R = \bigcup_{n=1}^{\infty} V_n$. Then, for an arbitrary Borel set B, let $B_1 = B \cap V_1$ and for $n \geq 2$ let us define recursively $B_n = (B \cap V_n) - \bigcup_{i=1}^{n-1} B_i$. Then B_n are mutually disjoint Borel sets on which π is injective and $B = \bigcup_{n=1}^{\infty} B_n$. We define μ on R by $\mu(B) = \sum_{n=1}^{\infty} \tilde{\mu}(\pi(B_n))$. The definition is independent of the sequence (B_n), since if (B'_m) is another sequence with the same properties, then

$$\begin{aligned} \sum_{n=1}^{\infty} \tilde{\mu}(\pi(B_n)) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tilde{\mu}(\pi(B_n \cap B'_m)) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{\mu}(\pi(B_n \cap B'_m)) \\ &= \sum_{m=1}^{\infty} \tilde{\mu}(\pi(B'_m)). \end{aligned}$$

Obviously, μ is h-invariant, since if π is injective on B_n and therefore on $h(B_n)$, then $\pi(B_n) = \pi(h(B_n))$ and $\tilde{\mu}(\pi(B_n)) = \tilde{\mu}(\pi(h(B_n))$ and consequently:

$$\mu(B) = \sum_{n=1}^{\infty} \tilde{\mu}(\pi(B_n)) = \sum_{n=1}^{\infty} \tilde{\mu}(\pi(h(B_n))) = \mu(h(B)) .$$

The fact that μ is a Borel measure and the uniqueness of μ are obvious.

Theorem 3 *If (R, h) is the orientable double cover of a nonorientable Klein surface S, then there is a Borel set $A \subseteq R$ such that:*

(a) $A \cap h(A) = \emptyset$
(b) $R = A \cup h(A)$
(c) the canonical projection $\pi : R \to S$ is injective and onto on both A and $h(A)$.

Proof: Let V_n be open sets such that π is injective on every V_n and $R = \bigcup_{n=1}^{\infty} V_n$. Let $B_1 = V_1$ and let us define recursively for every $n \geq 2$, $B_n = V_n - \bigcup_{i=1}^{n-1} B_i$. Then B_n are disjoint Borel sets, π is injective on every B_n and $R = \bigcup_{n=1}^{\infty} B_n$. Let $A_1 = B_1$ and for $n \geq 2$, let $A_n = B_n - h\left(\bigcup_{i=1}^{n-1} B_i\right)$, $A'_n = B_n \cap h\left(\bigcup_{i=1}^{n-1} B_i\right)$, $A = \bigcup_{n=1}^{\infty} A_n$, $A' = \bigcup_{n=2}^{\infty} A'_n$. Obviously, $A \cap A' = \emptyset$ and $A \cup A' = R$. Since $A_n \subseteq B_n$ for every n, it follows that A_n are mutually disjoint Borel sets and π is injective on A_n for every n. The same is true for A'_n. Let us show that π is injective on A and on A'. Suppose a_1, $a_2 \in A$, $a_1 \neq a_2$ and $\pi(a_1) = \pi(a_2)$. Then, there are m, $n \in \mathbb{N}$ such that $a_1 \in A_n$, $a_2 \in A_n$ and therefore $A_n \cap h(A_m) \neq \emptyset$. There is no loss of generality assuming that $m < n$. Then

$$A_n = B_n - h\left(\bigcup_{i=1}^{n-1} B_i\right) \subseteq B_n - h(B_m) \subseteq B_n - h(A_m), \text{ therefore}$$

$$A_n \cap h(A_m) \subseteq h(A_m) \cap [B_n - h(A_m)] = \emptyset, \text{ which is a contradiction.}$$

Suppose now that for $m < n$, there are $a_1 \in A'_m$, $a_2 \in A'_n$ such that $\pi(a_1) = \pi(a_2)$. Then $a_1 \in h(B_\kappa)$ for a $\kappa \leq m - 1$ and $h(a_1) \in B_\kappa$. Since $B_\kappa \cap B_n = \emptyset$, it follows that there are three distinct points in R: a_1, a_2 and $h(a_1)$ over the same point in S, which is a contradiction. Consequently, π is injective in each one of A and A'. Moreover, $A \cup A' = R$ and therefore $h(A) \subseteq A'$ and $h(A') \subseteq A$. By the fact that h is an involution, this last inclusion implies $A' \subseteq h(A)$ and therefore $A' = h(A)$. Since π is injective on both A and $h(A)$ and since $\pi(A \cup h(A)) = S$ and $\pi \circ h = \pi$, it follows that π is necessarily onto on both A and $h(A)$. We will call the couple $(A, h(A))$ a *covering partition* of R.

Theorem 4 *For any covering partition $(A, h(A))$ of R and any h-invariant Borel measure μ on R, the set function $\tilde{\mu}$ defined on the Borel sets of S by*

$$\tilde{\mu}(B) = \mu(\pi^{-1}(B) \cap A) = \mu(\pi^{-1}(B) \cap h(A)) \tag{3}$$

is a Borel measure on S.

Proof: It is obvious that if B is a Borel set on S, then $\pi^{-1}(B)$ is Borel set on R and so are the sets $\pi^{-1}(B) \cap A$ and $\pi^{-1}(B) \cap h(A)$. Consequently, the two last terms in (3) are well defined. The second equality in (3) is obvious since

$$\pi^{-1}(B) \cap h(A) = h(\pi^{-1}(B)) \cap h(A) = h(\pi^{-1}(B) \cap A)$$

and μ is h-invariant. Moreover, $\tilde{\mu}$ is a nonnegative set function which cancels on the empty set due to the same properties of μ and if B_n are disjoint Borel sets in S, then $\pi^{-1}(B_n) \cap A$ are disjoint Borel sets in R and

$$\begin{aligned} \tilde{\mu}\left(\bigcup_{n=1}^{\infty} B_n\right) &= \mu\left(\pi^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) \cap A\right) = \mu\left(\bigcup_{n=1}^{\infty}(\pi^{-1}(B_n) \cap A)\right) \\ &= \sum_{n=1}^{\infty} \mu(\pi^{-1}(B_n) \cap A) = \sum_{n=1}^{\infty} \tilde{\mu}(B_n). \end{aligned}$$

4. Potentials on Klein Surfaces

The potential p^{μ} generated by the measure μ on R is known (see [6]) to be a nonnegative superharmonic function on R, which is harmonic on the complementary set of the support of μ. Moreover, if μ is h-invariant, then:

$$p^{\mu}(a) = \int g_a(b)d\mu(b) = \int g_{h(a)}(h(b))d\mu(h(b)) = p^{\mu}(h(a)).$$

Therefore p^{μ} is h- invariant. On the other hand, if $(A, h(A))$ is a covering partition of R, then:

$$\begin{aligned} p^{\mu}(a) &= \frac{1}{2}\left[\int g_a d\mu + \int g_{h(a)} d\mu\right] = \int g_{\langle a \rangle} d\mu = \int_A g_{\langle a \rangle} d\mu \\ &+ \int_{h(A)} g_{\langle a \rangle} d\mu = 2\int g_{\tilde{a}} d\tilde{\mu} = 2p^{\tilde{\mu}}(\tilde{a}). \end{aligned}$$

It follows that the potential generated by $\tilde{\mu}$ is a nonnegative superharmonic function on S, which is harmonic on the complementary set of the support of $\tilde{\mu}$.

Theorem 5 *If $\tilde{f}$ is a function of class C_0^2 on S, then for every measure $\tilde{\mu}$ on S,*

(4) $$\int \tilde{f} d\tilde{\mu} = -\frac{1}{2\pi}\int p^{\tilde{\mu}} d^* d\tilde{f}.$$

Proof: Integrating (2) with respect to $\tilde{\mu}$ and using the Fubini's theorem we obtain (4).

It results that if $p^{\tilde{\mu}}$ is itself of class C^2, then for every function $\tilde{f} \in C_0(S)$:

(5) $$\int \tilde{f} d\tilde{\mu} = -\frac{1}{2\pi}\int \tilde{f} d^* dp^{\tilde{\mu}}.$$

Indeed, the Green's formula for $\tilde{f} \in C_0^2$ gives

$$\int \tilde{f} d^* dp^{\tilde{\mu}} = \int p^{\tilde{\mu}} d^* d\tilde{f}$$

and if $\tilde{f} \in C_0$, then we can use a sequence of C_0^2 functions converging uniformly to $\tilde{f}$.

A corollary of formula (5) is that if the difference of two potentials $p^{\tilde{\mu}}$ and $p^{\tilde{\nu}}$ is harmonic on an open set $G \subseteq S$, then $\tilde{\mu}$ and $\tilde{\nu}$ coincide in G. Indeed,

$$\int \tilde{f} d\tilde{\mu} - \int \tilde{f} d\tilde{\nu} = \frac{1}{2\pi} \int (p^{\tilde{\nu}} - p^{\tilde{\mu}}) d^* d\tilde{f} = \frac{1}{2\pi} \int \tilde{f} d^* d(p^{\tilde{\nu}} - p^{\tilde{\mu}}) = 0$$

for every $\tilde{f} \in C_0$ with the support in G, hence $\tilde{\mu}$ and $\tilde{\nu}$ coincide in G. It results that if $p^{\tilde{\mu}}$ is harmonic in G, then $\tilde{\mu}(G) = 0$.

A measure of a particular interest is the harmonic measure of a hyperbolic domain G. Since S is a locally compact space and the functional $\tilde{f} \to H^G_{\tilde{f}}(\tilde{a})$ is linear and positive on $C_0(S)$ for every $\tilde{a} \in G$, the Riesz representation theorem ensures the existence of a unique Borel measure $\omega^G_{\tilde{a}}$ on S such that for every $\tilde{f} \in C_0(S)$

$$H^G_{\tilde{f}}(\tilde{a}) = \int \tilde{f} d\omega^G_{\tilde{a}}.$$

The measure $\omega^G_{\tilde{a}}$ is called the *harmonic measure* of G at $\tilde{a}$. It can be easily seen that the support of $\omega^G_{\tilde{a}}$ is contained in ∂G and that $\omega^G_{\tilde{a}}(S) \leq 1$.

Theorem 6 *For every domain $G \subseteq S$, for every $\tilde{a} \in G$ and for every Borel set $B \subseteq S$,*

$$2\omega^G_{\tilde{a}}(B) = \omega_a^{\pi^{-1}(G)}(\pi^{-1}(B)) = \omega_{h(a)}^{\pi^{-1}(G)}(\pi^{-1}(B)) \tag{6}$$

Proof: The functionals $f \to H_f^{\pi^{-1}(G)}(a)$ and $f \to H_f^{\pi^{-1}(G)}(h(a))$ coincide on h-invariant functions $f \in C_0(R)$. Therefore:

$$\int f d\omega_a^{\pi^{-1}(G)} = \int f d_{h(a)}^{\pi^{-1}(G)} \text{ for } f \in C_0(R).$$

Futhermore, for every such function f, there is a unique function $\tilde{f} \in C_0(S)$ such that $f = \tilde{f} \circ \pi$. If $(A, h(A))$ is a covering partition of R, then :

$$\int \tilde{f} dw^G_{\tilde{a}} = \int_A f d\omega_a^{\pi^{-1}(G)} = \int_{h(A)} f d\omega_a^{\pi^{-1}(G)} = \frac{1}{2} \int f d\omega_a^{\pi^{-1}(G)}.$$

If $U \subseteq S$ is an open set, there is an increasing sequence of functions $\tilde{f}_n \in C_0(S)$ converging to χ_U. Then $f_n = \tilde{f}_n \circ \pi \in C_0(R)$ and the sequence (f_n) is increasing and convergent to $\chi_{\pi^{-1}(U)}$. Consequently:

$$\begin{aligned} \int \chi_U d\omega^G_{\tilde{a}} &= \frac{1}{2} \int \chi_{\pi^{-1}(U)} d\omega_a^{\pi^{-1}(G)}, \quad \text{or} \\ \omega^G_{\tilde{a}}(U) &= \frac{1}{2} \omega_a^{\pi^{-1}(G)}(\pi^{-1}(U)). \end{aligned}$$

This last equality and the regularity of the Borel measures $\omega^G_{\tilde{a}}$ and $\omega_a^{\pi^{-1}(G)}$ imply (6).

It is known (see [6]) that given a hyperbolic domain $V \subseteq R$ and a boundary function f, the function f is then and only then *resolutive* (for the Dirichlet problem) when $a \in V$ exists such that f is ω^V_a integrable. If such is the case, then f is ω^V_a integrable for every $a \in V$ and

$$H^V_f(a) = \int f d\omega^V_a$$

It results that for a hyperbolic domain $G \subseteq S$ a boundary function $\tilde{f}$ is resolutive if and only if $f = \tilde{f} \circ \pi$ is resolutive for $\pi^{-1}(G)$ and in the affirmative case

$$H_{\tilde{f}}^{G}(\tilde{a}) = H_{f}^{\pi^{-1}(G)}(a).$$

5. Compactifications of Nonorientable Klein Surfaces

Let S be an open nonorientable Klein surface and let (R, h) be its orientable double cover. We will be looking for a compactification R^* of R such that h admits an extension h^* to R^* and the couple (R^*, h^*) is a symmetric compact Riemann surface. Then, $S^* = R^*/\langle h^* \rangle$ is expected to be a compactification of S. Moreover, if $R^* - R$ is the ideal boundary of R in the compactification R^*, then $(R^* - R)/\langle h^* \rangle$ is expected to be the ideal boundary of S in the compactification S^*.

Different compactifications of open Riemann surfaces serve different purposes. A comprehensive presentation of compactifications of Riemann surfaces and of their purposes is given in [6].

Of special interest to this paper is the Kerékjártó-Stoilow (K.-S.) compactification [10], which is related to the Dirichlet problem [9]. We will use here both, the modern terminology [1] and the original terminology of Kerékjártó and Stoilow. Following Ahlfors and Sario, an *ideal boundary component* (or *element*) of R is a nonvoid collection q of domains of R which are not relatively compact on R, but have a compact boundary and which satisfy the following conditions:

(a) If $Q_0 \in q$ and $Q_0 \leq Q$, then $Q \in q$.
(b) If Q_1, $Q_2 \in q$, there exists a $Q_3 \subseteq Q_1 \cap Q_2$ which belongs to q.
(c) The intersection of all closures $\bar{Q}$, $Q \in q$ is empty.

For a given Q we denote by $\beta(Q)$ the set of all $q \in R^* - R$ such that $Q \in q$, therefore the statements $Q \in q$ and $q \in \beta(Q)$ are equivalent.

Theorem 7 *Let (R, h) be an open symmetric Riemann surface and let R^* be the K.-S. compactifcation of R. Then, h can be extended by continuity to an involution of R^*.*

Proof: It can be easily checked that for every $q \in R^* - R$, the collection $\{h(Q) : Q \in q\}$ fulfills the properties a) - c) listed above. Therefore, it defines a boundary element q^*. We put, by definition, $h^*(q) = q^*$ and let h^* be equal to h on R. We notice that $Q \in q$ is equivalent to $h(Q) \in q^*$ and this implies that $h^*(q^*) = q$. By consequence, h^* is an involution.

Let us show that h^* is continuous on R^*. There is no loss of generality by taking instead of the collection $\{Q : Q \in q\}$, a sequence (D_n) of domains of R fulfilling the following properties:

(I) Every D_n has a boundary which is a simple closed curve.
(II) $D_{n+1} \subseteq D_n$ for every n.
(III) $\bigcap_{n=1}^{\infty} \bar{D}_n = \emptyset$.

We say that (D_n) is a *determinant sequence* of q.

This was the original definition of a boundary element by Kerékjártó [8] and it has the advantage of allowing a simple definition of the convergence of a sequence (a_n) from R to an element $q \in R^* - R$. Namely, $\lim_{n\to\infty} a_n = q$ if and only if $a_n \in D_n$, where (D_n) is a determinant sequence of q.

Since (D_n) is a determinant sequence of q if and only if $(h(D_n))$ is a determinant sequence of $h^*(q)$, and since $a_n \in D_n$ if and only if $h(a_n) \in h(D_n)$, it follows that $\lim_{n\to\infty} a_n = q$ if and only if $\lim h^*(a_n) = h^*(q)$.

It remains to be shown that if (q_n) is a sequence in $R^* - R$ and $\lim_{n\to\infty} q_n = q$, then $\lim h^*(q_n) = h^*(q)$.

Let (D_i^n) be a determinant sequence of q_n and let (D_p) be a determinant sequence of q. We have $\lim_{n\to\infty} q_n = q$ if and only if, for every $p \in \mathbb{N}$, there is $n_p \in \mathbb{N}$ and there is $i_n \in \mathbb{N}$ such that $n \geq n_p$ and $i \geq i_n$ implies $D_i^n \subseteq D_p$. Then, $n \geq n_p$ and $i \geq i_n$ implies $h^*(D_i^n) \subseteq h^*(D_p)$ and therefore $\lim_{n\to\infty} h^*(q_n) = h^*(q)$.

In general it might happen that h^* has fixed points on $R^* - R$ even if h has been fixed-point free. This question will be dealt with in the following two theorems.

Theorem 8 *If (R, h) is an open symmetric Riemann surface of finite genus and R^* is the K.-S. compactification of R, then h can be extended by continuity to a fixed-point free involution of R^*.*

Proof: Let p be the genus of R. There is a compact set $K \subseteq R$ of genus p. For every element q on the ideal boundary of R, we can take determinant sequences (D_n) of q disjoint of K. Then D_n have all genus zero. Furthermore, since the ideal boundary of R is totally disconnected (see [1]), D_n can be taken such that any closed curve in D_n is either null homotopic, or homotopic to the boundary of D_n. Suppose that (D_n) and $(h(D_n))$ are determinant sequences of the same ideal boundary element of R. Then, for every n there is m_n such that $h(D_m) \subseteq D_n$ for $m \geq m_n$. There is a conformal mapping φ of $h(D_n)$ on the exterior of the unit disk. Then $\varphi \circ h \circ \varphi^{-1}$ is an anticonformal mapping of the exterior of the unit disk on itself and therefore is the complex conjugate of a Möbius transformation with non constant denominator. By consequence, there is z_0 such that $1 < |z_0|$ and $(\varphi \circ h \circ \varphi^{-1})(z_0) = z_0$, or $h \circ \varphi^{-1}(z_0) = \varphi^{-1}(z_0)$. Thus, h has a fixed point, which is a contradiction. Therefore, (D_n) and $(h(D_n))$ cannot define the same ideal boundary element of R, which means that $h^*(q) \neq q$ for every $q \in R^* - R$, and that h^* is a fixed-point free involution.

Theorem 9 *Let (R, h) be a symmetric Riemann surface. Then there is on R an antianalytic involution h_1 that can be extended by continuity to the K.-S. compactification R^* of R such that the extension h_1^* is a fixed-point free antianalytic involution of R^*. Moreover, $R^*/\langle h_1^* \rangle$ is the K.-S. compactifcation of $R/\langle h \rangle$.*

Proof: Let $S = R/\langle h \rangle$ and let S^* be the K.-S. compactification of S. There is a unique (up to a conformal mapping) compact Riemann surface R^* and there is a fixed-point free antianalytic involution h_1^* of R^* such that $R^*/\langle h_1^* \rangle = S^*$.

Let π be the canonical projection of R^* on S^*. Then $\pi^{-1}(S)$ is the orientable double cover of S and therefore it is conformally equivalent with R. We may consider them identical and take $h_1 = h_{1|R}^*$.

We want to show that R^* is the K.-S. compactification of R. It is known (see [1]) that there is a unique compactification of R which is a locally connected Hausdorff space and whose ideal boundary is totally disconnected and nonseparating. We will show that R^* verifies these properties.

Obviously R^* is a locally connected Hausdorff space. Suppose $q \in R^* - R$ and let $p = \pi(q) = \pi(h_1^*(q))$. It is also evident that $p \in S^* - S$. Given a sequence (a_n) in S with limit p, the set $\{\pi^{-1}(\{a_n\})\}$ consists of exactly two points for every n. The set of all these points is an infinite set on the compact space R^*. Therefore it has at least one cluster point. Taking into account the fact that π is a local homeomorphism, one of these cluster points must be q. It follows that R is dense in R^*. The fact that $R^* - R$ is totally disconnected is obvious, since any connected component of $R^* - R$ would project by π into a connected component of $S^* - S$. However, $S^* - S$ has no connected component.

It remains to be shown that $R^* - R$ is nonseparating on R^*. Suppose $G \subseteq R^*$ is an open connected set such that $G \cap R = O_1 \cup O_2$ where O_1 and O_2 are nonempty open disjoint sets. Then $\pi(G)$ is an open connected set and

$$\pi(G \cap R) = \pi(G) \cap S = \pi(O_1) \cup \pi(O_2) ,$$

where $\pi(O_1)$ and $\pi(O_2)$ are non empty open disjoint sets.

Since $U \in p$ is equivalent to $p \in \beta(U)$, it results that $\beta(\pi(O_1)) \cap \beta(\pi(O_2)) = \emptyset$. Obviously, $\pi(O_i) \cap \beta(\pi(O_j)) = \emptyset$ for $i \neq j$, therefore

$$[\pi(O_1) \cup \beta(\pi(O_1))] \quad \cap \quad [\pi(O_2) \cup \beta(\pi(O_2))] = \emptyset, \text{ and}$$
$$\pi(G) = [\pi(O_1) \cup \beta(\pi(O_1))] \quad \cup \quad [\pi(O_2) \cup \beta(\pi(O_2))],$$

which contradicts the fact that $\pi(G)$ is connected. Consequently, $R^* - R$ is nonseparating on R^*.

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