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STARLIKE FUNCTIONS IN AN ELLIPTICAL DOMAIN

NICOLAE N. PASCU DORINA RADUCANU
RADU N. PASCU MIHAI N. PASCU

1. Introduction

Let g be a complex function in the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$. For $z = x + iy$ we put $u(x,y) = \operatorname{Re} g(z)$ and $v(x,y) = \operatorname{Im} g(z)$. The function g belongs to the class $C^1(U)$ if the functions $u = u(x,y)$ and $v = v(x,y)$ are continuous and have continuous first order partial derivatives in U.If $g \in C^1(U)$ we denote

$$Dg = z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}} \qquad \text{and} \qquad Jg = \left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \bar{z}}\right|^2$$

where

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right).$$

In 1980, P.T. Mocanu [1], introduced the class of nonanalytic starlike functions in U and obtained sufficient conditions for complex nonanalytic functions in U to be starlike.

A continuous function $f : U \to \mathbf{C}, f(0) = 0$ is starlike in U if it is univalent and the range $f(U)$ is starlike with respect to the origin.

Theorem 1 [1] *If the function g belongs to the class $\mathrm{C}^1(U)$ and satisfies the following conditions:*

(i) $g(0) = 0$ and $g(z) \neq 0$, for all $z \in U \setminus \{0\}$

(ii) $Jg(z) > 0$, for all $z \in U$

(iii) $\operatorname{Re}\frac{Dg(z)}{g(z)} > 0$, for all $z \in U \setminus \{0\}$

then g is starlike in U.

2. Starlikeness conditions

An analytic function $f : E \to \mathbf{C}, f(0) = 0$ is called *starlike in* E if it is univalent in E and $f(E)$ is a starlike domain with respect to the origin.

The following theorems provide sufficient conditions of starlikeness.

Theorem 1 *If the function f is an analytic function from E into $\mathbf{C}$ and satisfies the conditions:*

(i) $f(0) = 0, f(z) \neq 0$, for all $z \in E \setminus \{0\}$ and $f'(z) \neq 0$, for all $z \in E$

(ii) For any $z \in E$, the following inequality, holds

$$(a^2 + b^2) \operatorname{Re} \frac{zf'(z)}{f(z)} - (a^2 - b^2) \operatorname{Re} \frac{\bar{z}f'(z)}{f(z)} > 0 \tag{1}$$

then f is a starlike function in E.

Proof. Let h be the function from U into E given by

$$h(z) = \frac{a+b}{2} z + \frac{a-b}{2} \bar{z}. \tag{2}$$

Then $h \in C^1(U)$, h is injective in U and $h(U) = E$. We consider the function $g : U \to \mathbf{C}, g = f \circ h$. We shall prove that if the function f satisfies the conditions $(i) - (ii)$ of Theorem 2, then the function g satisfies the conditions of Theorem 1. Hence g is a starlike function in U and since $f(E) = g(U)$ we obtain that f is a starlike function in E.

We have $g(z) = f\left(\frac{a+b}{2} z + \frac{a-b}{2} \bar{z}\right) \in C^1(U), g(0) = f(0) = 0$ and $g(z) \neq 0$, for all $z \in U \setminus \{0\}$. We also have

$$Jg(z) = \left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \bar{z}}\right|^2 = ab\left|f'(u)\right|^2 > 0,$$

where $u = h(z) \in E$.

By using the definition of the operator D, we obtain

$$\frac{Dg(z)}{g(z)} = \frac{\left(\frac{a+b}{2} z - \frac{a-b}{2} \bar{z}\right) f'(u)}{f(u)}. \tag{3}$$

From $u = h(z) = \frac{a+b}{2} z + \frac{a-b}{2} \bar{z}$ and $\bar{u} = \frac{a-b}{2} z + \frac{a+b}{2} \bar{z}$ it results

$$z = \frac{1}{2ab} \left[(a+b) u - (a-b) \bar{u}\right] \tag{4}$$

From (1),(3) and (4), we obtain

$$\operatorname{Re} \frac{Dg(z)}{g(z)} = (a^2 + b^2) \operatorname{Re} \frac{uf'(u)}{f(u)} - (a^2 - b^2) \operatorname{Re} \frac{\bar{u}f'(u)}{f(u)} > 0.$$

Remark. For $a = b$, we have $E = U$ and we obtain the well known condition of starlikeness for analytic functions in U.

Theorem 2 *Let f be an analytic function from E into $\mathbf{C}$. If f satisfies the following conditions:*

(i) $f(0) = 0, f(z) \neq 0$ for all $z \in E \setminus \{0\}$ and $f'(z) \neq 0$, for all $z \in E$

(ii) For any $z \in E$

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \arccos \frac{a^2 - b^2}{a^2 + b^2} \tag{5}$$

then f is a starlike function in E.

Proof. In order to prove that f is a starlike function in E, we shall show that the inequality (1) holds. Since

$$-\left(a^2 - b^2\right) \operatorname{Re} \frac{\bar{z}f'(z)}{f(z)} \geq -\left(a^2 - b^2\right) \left|\frac{zf'(z)}{f(z)}\right| \qquad , \qquad \text{for all } z \in E$$

we obtain

$$\left(a^2 + b^2\right) \operatorname{Re} \frac{zf'(z)}{f(z)} - \left(a^2 - b^2\right) \operatorname{Re} \frac{\bar{z}f'(z)}{f(z)} \geq$$

$$\geq \left(a^2 + b^2\right) \operatorname{Re} \frac{zf'(z)}{f(z)} - \left(a^2 - b^2\right) \frac{zf'(z)}{f(z)} =$$

$$= \left(a^2 + b^2\right) \left|\frac{zf'(z)}{f(z)}\right| \left\{ \frac{\operatorname{Re} \frac{zf'(z)}{f(z)}}{\left|\frac{zf'(z)}{f(z)}\right|} - \frac{a^2 - b^2}{a^2 + b^2} \right\} =$$

$$\left(a^2 + b^2\right) \left|\frac{zf'(z)}{f(z)}\right| \left\{ \cos \left[\arg \frac{zf'(z)}{f(z)}\right] - \frac{a^2 - b^2}{a^2 + b^2} \right\} > 0.$$

Hence, f is a starlike function in E.

References

[1] Mocanu, P. T., Starlikeness and convexity for non-analytic functions in the unit disk, Mathematica, Tome 22(45), no1(1980),77-83.

[2] Royster, W.C., Convexity and starlikeness of analytic functions, Duke. Math. 19(1952), 447-457.