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THE CAUCHY-IONESCU PROBLEM FOR HYPERBOLIC INCLUSIONS WITH MODIFIED ARGUMENT

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Abstract. In this paper we consider the Cauchy- Ionescu Problem for hyperbolic inclusions with modified argument. An existence theorem for a local solution of this problem is proved and some properties of the set of its solutions are established.

Key words: multifunction, hyperbolic inclusion, upper-semicontinuity, initial values, absolutely continuous in Carathéodory's sense function.

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1 Introduction

In his PhD Thesis (1927) [10], D.V. Ionescu studied — for the first time in mathematical literature — boundary value problems of Darboux, Picard, Cauchy and Goursat types for second order partial differential equations with modified argument.

More recently, a series of authors studied the same problems for second order hyperbolic equations of various forms.

In this paper we consider the Cauchy-Ionescu Problem for hyperbolic inclusions with modified argument of the form

$$\frac{\partial^2 z(x,y)}{\partial x \partial y} \in F(x,y,z(g(x,y),h(x,y))), \quad (x,y) \in D = [0,a] \times [0,b], \tag{1.1}$$

$$\begin{cases} z(x,\varphi(x)) = R(x), & 0 \le x \le a, \\ \dfrac{\partial z}{\partial x}(x,\varphi(x)) = P(x), & 0 \le x \le a, \end{cases} \tag{1.2}$$

where the curve $\gamma : y = \varphi(x)$, $\varphi \in C^1([0,a];[0,b])$ is given, with φ' absolutely continuous and

$$\varphi(0) = 0, \quad \varphi(a) = b, \quad \varphi'(x) > 0, \quad 0 \le x \le a, \tag{1.3}$$

$F : D \times \Omega \to 2^{\mathbb{R}}$ is a multifunction with compact, convex and non-empty values, Ω is an open subset of $\mathbb{R}^n$, $g \in C(D;[0,a])$, $h \in C(D;[0,b])$, $R \in AC([0,a];\mathbb{R}^n)$, $P \in AC([0,b];\mathbb{R}^n)$.

Under suitable assumptions, we prove an existence theorem for a local solution of this problem, and that the set of solutions is compact in Banach space $C(D_0;\mathbb{R}^n)$, $D_0 = [0,x_0] \times [0,y_0] \subseteq D$; moreover, as a function of the initial values, this set defines an upper-semicontinuous multifunction.

This study was suggested by papers which deal with the Cauchy-Ionescu Problem for hyperbolic equations [6], [7] and [15].

Remarks.

1. The conditions (1.3) can be changed by

$$\varphi(0) = b, \quad \varphi(a) = 0, \quad \varphi'(x) < 0, \quad 0 \le x \le a. \tag{1.4}$$

2. The relation $\varphi'(x) > 0$ (or $\varphi'(x) < 0$), for $0 \le x \le a$, shows that the curve γ is not tangent to any of the "characteristics" $x = 0$, $y = 0$.
3. In view of relation $\varphi'(x) > 0$, if φ satisfies conditions (1.3), the equation of curve γ can written as

$$\gamma : x = \psi(y), \quad 0 \le y \le b, \quad \psi = \varphi^{-1}, \quad \psi \in C^1([0,b];[0,a]),$$

ψ' is absolutely continuous on $[0,b]$, with the conditions

$$\psi(0) = 0, \quad \psi(b) = a, \quad \psi'(y) > 0, \quad 0 \le y \le b.$$

4. From conditions (1.2), it follows that the values of the derivative $\dfrac{\partial z}{\partial y}$ on curve γ are known.

Indeed, differentiating the first of equations (1.2) with respect to x, under hypotheses allowing this operation, we obtain

$$\frac{\partial z}{\partial x}(x,\varphi(x)) + \frac{\partial z}{\partial y}(x,\varphi(x))\varphi'(x) = R'(x), \quad 0 \le x \le a,$$

whence it results

$$\frac{\partial z}{\partial y}(x,\varphi(x)) = \frac{1}{\varphi'(x)}\left[R'(x) - \frac{\partial z}{\partial x}(x,\varphi(x))\right] = \frac{1}{\varphi'(x)}[R'(x) - P(x)], \quad 0 \le x \le a.$$

We denote

$$Q(x) = \frac{1}{\varphi'(x)}[R'(x) - P(x)], \quad 0 \le x \le a. \tag{1.5}$$

From our hypotheses on the functions φ, R, P, it results that function Q is absolutely continuous, $Q \in AC([0,a];\mathbb{R}^n)$.

Taking into account this remark, the following conditions can be taken instead of conditions (1.2):

$$\begin{cases} z(x,\varphi(x)) = R(x), & 0 \le x \le a, \\ \dfrac{\partial z}{\partial y}(x,\varphi(x)) = Q(x), & 0 \le x \le a, \end{cases} \tag{1.6}$$

or

$$\begin{cases} z(x,\varphi(x)) = R(x), & 0 \le x \le a, \\ \dfrac{\partial z}{\partial x}(x,\varphi(x)) = P(x), & 0 \le x \le a, \\ \dfrac{\partial z}{\partial y}(x,\varphi(x)) = Q(x), & 0 \le x \le a. \end{cases} \tag{1.7}$$

5. The Cauchy-Ionescu Problem may be also restated under other conditions, namely

$$\begin{cases} z(x,\varphi(x)) = R(x), & 0 \le x \le a, \\ \dfrac{\partial z}{\partial n}(x,\varphi(x)) = S(x), & 0 \le x \le a, \end{cases} \tag{1.8}$$

where $\dfrac{\partial z}{\partial n}$ is the derivative along the normal direction to curve γ, and $S \in AC([0,a];\mathbb{R}^n)$. Indeed, the knowledge of the functions z and $\dfrac{\partial z}{\partial n}$ on curve γ is equivalent to knowing the functions z and $\dfrac{\partial z}{\partial x}$ $\left(\text{or } \dfrac{\partial z}{\partial y}\right)$ on γ, due to equations [16, p. 78 and p. 268]

$$\begin{cases} cos(\widehat{n,x}) = \dfrac{\varphi'}{\sqrt{1+(\varphi')^2}}, \qquad cos(\widehat{n,y}) = -\dfrac{1}{\sqrt{1+(\varphi')^2}}, \\ \dfrac{\partial z}{\partial x}(x,\varphi(x)) + \dfrac{\partial z}{\partial y}(x,\varphi(x)) \cdot \varphi'(x) = R'(x), \quad 0 \le x \le a, \\ \dfrac{\partial z}{\partial x}(x,\varphi(x)) \cdot \dfrac{\varphi'}{\sqrt{1+(\varphi')^2}} - \dfrac{\partial z}{\partial y}(x,\varphi(x)) \cdot \dfrac{1}{\sqrt{1+(\varphi')^2}} = S(x), \quad 0 \le x \le a. \end{cases}$$

2 Preliminaries

The definitions and Theorem 2.1 in this section are taken from [1], [4], [5], [11]–[14].

Definition 2.1. *Let X and Y be two non-empty sets. A multifunction $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.*

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup_{x\in X} \Phi(x)$ is the *range* of Φ.

Definition 2.2. *Let us consider $\Phi : X \to 2^Y$.*

a) If $A \subset X$, the image of A by Φ is $\Phi(A) = \bigcup_{x\in A} \Phi(x)$;
b) If $B \subset Y$, the counterimage of B by Φ is $\Phi^-(B) = \{x \in X | \Phi(x) \cap B \neq \emptyset\}$;
c) The graph of Φ, denoted graph Φ is the set

$$graph\ \Phi = \{(x,y) \in X \times Y | y \in \Phi(x)\}.$$

Definition 2.3. *Let now take $\Phi : X \to 2^X$. An element $x \in X$ with the property $x \in \Phi(x)$ is called a fixed point of the multifunction Φ.*

Definition 2.4. *A univalued function $\varphi : X \to Y$ is said to be a selection of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.*

Definition 2.5. *Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is upper-semicontinuous if, for any closed subset $B \subset Y$, $\Phi^-(B)$ is closed in X.*

Definition 2.6. *If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is measurable if $\Phi^-(B) \in \mathcal{F}$ for every closed subset $B \subset Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.*

Theorem 2.1. [14] *Let X and Y be two compact metric spaces and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

- (*i*) *the multifunction Φ is upper-semicontinuous;*
- (*ii*) *the graph of Φ is a closed subset of $X \times Y$;*
- (*iii*) *any would be the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi(x_n)$, $y_n \to y$ it follows $y \in \Phi(x)$.*

Definition 2.7. [4], [5] *The function $u : D \to \mathbb{R}^n$ is absolutely continuous in Carathéodory's sense* [1, §565–§570] *iff $u(x, y)$ is continuous on D, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x, y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue-integrable on D.*

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(D; \mathbb{R}^n)$ [4], [5].

We denote by $AC([t_1, t_2]; \mathbb{R}^n)$ the space of absolutely continuous functions $f : [t_1, t_2] \to \mathbb{R}^n$, endowed with the norm

$$\|f\| = \sup_{t\in[t_1,t_2]} \|f(t)\| + \int_{t_1}^{t_2} \|f'(t)\| dt.$$

3 Results

In a similar way as in [2] and [15], we define the notion of a *local solution* for the Cauchy-Ionescu Problem (1.1)–(1.2) and we prove an existence theorem for a local solution of this problem, together with some properties of the set of solutions, namely that this is a compact subset in Banach space $C(D_0; \mathbb{R}^n)$ and, as a function of initial values, it defines an upper-semicontinuous multifunction.

Let the following hypotheses be satisfied:

- (H_0) The curve $\gamma : y = \varphi(x)$, $0 \le x \le a$, is defined by the function $\varphi \in C^1([0, a]; [0, b])$ with $\varphi' \in AC([0, a]; \mathbb{R})$ and properties (1.3);
- (H_1) $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex, non-empty values in $\mathbb{R}^n$, $D = [0, a] \times [0, b] \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^n$ is an open subset;
- (H_2) For any $(x, y) \in D$ the mapping $z \to F(x, y, z)$ is upper-semicontinuous on Ω;
- (H_3) For any $z \in \Omega$ the mapping $(x, y) \to F(x, y, z)$ is Lebesgue measurable on D;
- (H_4) $g \in C(D; [0, a])$ and $h \in C(D; [0, b])$;
- (H_5) There exists a function $k : D \to \mathbb{R}_+$, $k \in \mathcal{L}^1(D; \mathbb{R}_+)$ such that

$$\|\zeta\| \le k(x, y) \quad \text{for} \quad \forall\zeta \in F(x, y, z), \quad \forall(x, y) \in D, \quad \forall z \in \Omega;$$

(H_6) There exists a convex, compact set $M \subset \Omega$ and a point $(x_0, y_0) \in]0,a]\times]0,b]$, $y_0 = \varphi(x_0)$, such that

$$\int_0^{x_0}\int_0^{y_0} k(u,v)du\, dv \le d(M, C_\Omega),$$

where $d(M, C_\Omega)$ is the distance from M to $C_\Omega = \mathbb{R}^n - \Omega$;

(H_7) $P, R, R' \in AC([0,a];\mathbb{R}^n)$;

(H_8) The values of function $\alpha : D \to \mathbb{R}^n$, defined by

$$\alpha(x,y) = R(x) - \int_y^{\varphi(x)} Q(\psi(v))dv, \tag{3.1}$$

belong to the set M for $(x,y) \in D_0 = [0,x_0] \times [0,y_0]$, where $Q : [0,a] \to \mathbb{R}^n$ is the function defined by (1.5).

Remarks.

6. It follows from hypotheses (H_0), namely $\varphi' \in AC([0,a];[0,b])$, and ($H_7$) stating that $R', P \in AC([0,a];\mathbb{R}^n)$, in view of operations with absolutely continuous functions, that the function $Q \in AC([0,a];\mathbb{R}^n)$.
7. It follows from hypotheses (H_0), namely $\varphi' \in AC([0,a];[0,b])$, and ($H_7$), stating that $R \in AC([0,a];\mathbb{R}^n)$, and also from the previous Remark, that the function α defined by (3.1) is absolutely continuous in Carathéodory's sense [1], $\alpha \in C^*(D;\mathbb{R}^n)$.

Definition 3.1. *The Cauchy-Ionescu Problem for the hyperbolic inclusion with modified argument* (1.1) *means to determine a solution of this inclusion which satisfies the initial conditions* (1.2).

Definition 3.2. *It is defined a local solution of the Cauchy-Ionescu Problem* (1.1) + (1.2) *as a function* $Z : D_0 \to \Omega$, $Z \in C^*(D_0;\mathbb{R}^n)$, *which is absolutely continuous is Carathéodory's sense* [1] *and satisfies* (1.1) a.e. *for* $(x,y) \in D_0$, *and also conditions* (1.2) *for all* $x \in [0,x_0]$.

Theorem 3.1. *Let the hypotheses* (H_0)–(H_8) *be satisfied. Then:*

(*i*) *there exists at least a local solution* Z *of the Cauchy-Ionescu Problem* (1.1) + (1.2);

(*ii*) *the set* S_α *of local solutions* Z *is compact in the Banach space* $C(D_0;\mathbb{R}^n)$;

(*iii*) *the multifunction* $\alpha \to S_\alpha$ *is upper-semicontinuous on* $C^*(D_0;\mathbb{R}^n)$ *taking values in* $C(D_0;\mathbb{R}^n)$.

PROOF. (*i*) Let $C^*(D_0;\mathbb{R}^n)$ be the set of absolutely continuous functions in Carathéodory's sense defined on D_0 with values in $\mathbb{R}^n$ [1]. We denote by $\mathcal{Z}_M$ the set of functions $Z : D_0 \to \mathbb{R}^n$, $Z \in C^*(D_0;\mathbb{R}^n)$, which satisfy the inequality

$$\left\| \frac{\partial^2 Z(x,y)}{\partial x \partial y} \right\| \le k(x,y), \quad \text{a.e. for} \quad (x,y) \in D_0, \tag{3.2}$$

and also conditions (1.2). The notation $\mathcal{Z}_M$ is suitable because, by hypothesis (H_8), $\alpha(x,y) \in M$ for $(x,y) \in D_0$. We remark that the absolute continuity in Carathéodory's sense of Z assures the existence of the derivative $\dfrac{\partial^2 Z(x,y)}{\partial x \partial y}$ a.e. for $(x,y) \in D_0$ [1, §565–§570].

We have $\mathcal{Z}_M \subset C^*(D_0;\mathbb{R}^n)$. Then, by hypothesis (H_6) and inequality (3.2), for any $Z \in \mathcal{Z}_M$, it follows that $Z(x,y) \in \Omega$.

Indeed, let $N(x,y)$, $N'(x=\psi(y),y)$, $N''(x,\varphi(x))$, with $y \leq \varphi(x)$ be three points in D and consider the triangle

$$T(x,y) = \{(u,v)|\psi(v) \leq u \leq x,\ y \leq v \leq \varphi(x)\} \subset D_0, \quad (x,y) \in D.$$

We have

$$\begin{aligned}\iint_{T(x,y)} \frac{\partial^2 Z(u,v)}{\partial u \partial v} du\,dv = \int_y^{\varphi(x)} dv \int_{\psi(v)}^{x} \frac{\partial^2 Z(u,v)}{\partial u \partial v} du &= \\ = Z(x,\varphi(x)) - Z(x,y) - \int_y^{\varphi(x)} Q(\psi(v))dv &= \\ = R(x) - Z(x,y) - \int_y^{\varphi(x)} Q(\psi(v))dv. & \qquad (3.3)\end{aligned}$$

It results

$$\begin{aligned}Z(x,y) = R(x) - \int_y^{\varphi(x)} Q(\psi(v))dv - \iint_{T(x,y)} \frac{\partial^2 Z(u,v)}{\partial u \partial v} du\,dv &= \\ = \alpha(x,y) - \iint_{T(x,y)} \frac{\partial^2 Z(u,v)}{\partial u \partial v} du\,dv. & \qquad (3.4)\end{aligned}$$

It follows that

$$\frac{\partial Z}{\partial x}(x,y) = R'(x) - Q(x) \cdot \varphi'(x) - \int_y^{\varphi(x)} \frac{\partial^2 Z(x,v)}{\partial x \partial v} dv. \qquad (3.5)$$

From (3.4) and (3.5), it results that the function Z defined by (3.4) satisfies conditions (1.2).

Using hypothesis (H_6), inequality (3.2) and (3.4) we obtain

$$\|Z(x,y) - \alpha(x,y)\| \leq \iint_{T(x,y)} k(u,v)du\,dv \leq \iint_{D_0} k(u,v)du\,dv \leq d(M, C_\Omega). \qquad (3.6)$$

From the hypotheses (H_8), $\alpha(x,y) \in M$ for $(x,y) \in D_0$ and we have

$$d(Z(x,y),\alpha(x,y)) = \|Z(x,y) - \alpha(x,y)\| \leq d(M, C_\Omega), \qquad (3.7)$$

which shows that $Z(x,y) \in \Omega$.

The set of functions $\mathcal{Z}_M$ is *convex* and *compact* in $C(D_0;\mathbb{R}^n)$. The convexity results by the definition of this set, and its compactness from the Arzelà-Ascoli theorem, using hypotheses (H_0), (H_6), (H_7), (H_8).

We denote by $\mathcal{G}$ the set of the triples $(\alpha, Z, U) \in C^*(D_0;\mathbb{R}^n) \times \mathcal{Z}_M \times \mathcal{Z}_M$ with the property that Z and U satisfy the membership relation

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} \in F(x,y,Z(g(x,y),h(x,y))), \quad \text{a.e. for} \quad (x,y) \in D_0. \qquad (3.8)$$

We prove that, for each $\alpha \in C^*(D_0;\mathbb{R}^n)$ with $\alpha(x,y) \in M$ for $(x,y) \in D_0$, the set of (Z,U) such that $(\alpha, Z, U) \in \mathcal{G}$ is non-empty and the set $\mathcal{G}$ is closed.

Indeed, let us take $Z \in Z_M$. From Theorem 1 [2], there exists a μ-measurable (under the μ-Lebesgue measure) multifunction $\Gamma : D_0 \to 2^{\mathbb{R}^n}$ with compact, non-empty values in $\mathbb{R}^n$ such that

$$\Gamma(x,y) \subset F(x,y,Z(g(x,y),h(x,y))), \quad \forall (x,y) \in D_0. \tag{3.9}$$

Then, by Theorem 2 or Theorem 3 [3], there exists a measurable selection β of Γ, i.e. a measurable univalued function $\beta : D_0 \to \mathbb{R}^n$ with $\beta(x,y) \in \Gamma(x,y)$ for $(x,y) \in D_0$.

Let the function $U : D_0 \to \mathbb{R}^n$ be defined by

$$U(x,y) = \alpha(x,y) - \iint_{T(x,y)} \beta(u,v)\,du\,dv, \quad (x,y) \in D_0. \tag{3.10}$$

Then, the set of those pairs (Z,U) such that $(\alpha, Z, U) \in \mathcal{G}$ is *non-empty* because

$$\beta(x,y) \in \Gamma(x,y) \subset F(x,y,Z(g(x,y),h(x,y))), \quad \text{a.e. for} \quad (x,y) \in D_0, \tag{3.11}$$

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} = \beta(x,y) \in \Gamma(x,y) \subset F(x,y,Z(g(x,y),h(x,y))), \ \text{a.e.}\ (x,y) \in D_0, \tag{3.12}$$

$$\left\| \frac{\partial^2 U(x,y)}{\partial x \partial y} \right\| = \|\beta(x,y)\| \le k(x,y), \quad (x,y) \in D_0, \tag{3.13}$$

by hypotheses (H_5) for $\zeta = \beta(x,y)$, and

$$\begin{cases} U(x,\varphi(x)) = R(x), & 0 \le x \le x_0, \\ \dfrac{\partial U}{\partial x}(x,\varphi(x)) = P(x), & 0 \le x \le x_0. \end{cases} \tag{3.14}$$

For the proof that $\mathcal{G}$ is closed, we consider a sequence of elements in $\mathcal{G}$, $\{(\alpha_n, Z_n, U_n)\}_{n \in \mathbb{N}}$, convergent to (α, Z, U) in the space $C^*(D_0;\mathbb{R}^n) \times C(D_0;\mathbb{R}^n) \times L^1(D_0;\mathbb{R}^n)$. We must check that $(\alpha, Z, U) \in \mathcal{G}$, what implies, by the definition of set $\mathcal{G}$, that conditions (1.2) and (3.12) are satisfied by Z and U.

The set $\left\{ \dfrac{\partial^2 U_n(x,y)}{\partial x \partial y} \right\}_{n \in \mathbb{N}}$ is *relatively weakly compact* in $L^1(D_0;\mathbb{R}^n)$ by the Dunford-Pettis Criterion [8]. It follows that $\left\{ \dfrac{\partial^2 U_n(x,y)}{\partial x \partial y} \right\}_{n \in \mathbb{N}}$ is weakly convergent to a function $V \in L^1(D_0;\mathbb{R}^n)$. For each $(x,y) \in D_0$, we have

$$\begin{aligned} U(x,y) &= w - \lim_{n\to\infty} U_n(x,y) = w - \lim_{n\to\infty} \left[\alpha_n(x,y) - \iint_{T(x,y)} \frac{\partial^2 U_n(u,v)}{\partial u \partial v} du\,dv \right] = \\ &= \alpha(x,y) - \iint_{T(x,y)} V(u,v)\,du\,dv. \end{aligned} \tag{3.15}$$

From the weak convergence $\dfrac{\partial^2 U_n(x,y)}{\partial x \partial y} \rightharpoonup V(x,y)$, $(x,y) \in D_0$, using the Corollary of Mazur's Theorem [9], it follows that there exists a sequence of convex combinations $\{W_r\}_{r \in \mathbb{N}}$

of the set $\left\{\frac{\partial^2 U_r}{\partial x\partial y}, \frac{\partial^2 U_{r+1}}{\partial x\partial y}, \dots\right\}$, strongly convergent to V in $L^1(D_0;\mathbb{R}^n)$. Then, we can extract a subsequence from the sequence $\{W_r\}_{r\in\mathbb{N}}$ which converges a.e. to V : $W_{r_i} \to V$ a.e. for $(x,y)\in D_0$.

Since $F(x,y,Z)$ is convex and compact for all $(x,y)\in D$ and for all $Z\in\Omega$, we obtain from the previous results and from Lemma 2 [2] that

$$\begin{aligned} V(x,y) \in \bigcap_{r=1}^{\infty} &\overline{conv\left(\bigcup_{n=r}^{\infty}\frac{\partial^2 U_n(x,y)}{\partial x\partial y}\right)} \subset \\ \subset \bigcap_{r=1}^{\infty} &\overline{conv\left(\bigcup_{n=r}^{\infty} F(x,y,Z_n(g(x,y),h(x,y)))\right)} \subset \\ \subset F(x,y,&Z(g(x,y),h(x,y))), \qquad \text{a.e. for } (x,y)\in D_0, \end{aligned} \tag{3.16}$$

from which it follows that $\mathcal{G}$ *is closed.*

Indeed, (3.16) shows that $V(x,y)\in F(x,y,Z(g(x,y),h(x,y)))$ a.e. for $(x,y)\in D_0$, and we obtain $\frac{\partial^2 U(x,y)}{\partial x\partial y} = V(x,y)$ from (3.15); then, using (3.8) and (3.16) we have

$$V(x,y) = \frac{\partial^2 U(x,y)}{\partial x\partial y} \in F(x,y,Z(g(x,y),h(x,y))), \qquad \text{a.e. for } (x,y)\in D_0, \tag{3.17}$$

and also (3.14); hence U satisfies initial conditions (1.2) for $0\le x\le x_0$.

Let us take $\alpha\in C^*(D_0;\mathbb{R}^n)$ with $\alpha(x,y)\in M$ for $(x,y)\in D_0$. To each $Z\in\mathcal{Z}_M$ we associate the set $\Phi(Z)\subset\mathcal{Z}_M$ as follows:

$$U\in\Phi(Z)\Leftrightarrow U\in\mathcal{Z}_M,\ \frac{\partial^2 U(x,y)}{\partial x\partial y}\in F(x,y,Z(g(x,y),h(x,y))), \quad \text{a.e. } (x,y)\in D_0. \tag{3.18}$$

We thus define a multifunction $\Phi:\mathcal{Z}_M\to 2^{\mathcal{Z}_M}$. The set $\Phi(Z)$ is *convex, compact* and *non-empty*. It can be seen that $\Phi(Z)$ is convex since $F(x,y,Z(x,y))$ is convex by hypothesis (H_1). We have $\Phi(Z)\subset\mathcal{Z}_M$ but $\mathcal{Z}_M$ is compact. The multifunction Φ has a closed graph because *graph* Φ is the set $\mathcal{G}$ for each fixed α and $\mathcal{G}$ is closed. It follows that $\Phi(Z)$ is compact in $C(D_0;\mathbb{R}^n)$ as a closed subset of the compact set $\mathcal{Z}_M$. The set $\Phi(Z)$ is non-empty since there exists U, defined by (3.10), with the property $U\in\Phi(Z)$.

The multifunction $\Phi:\mathcal{Z}_M\to 2^{\mathcal{Z}_M}$, having a closed graph, is upper-semicontinuous by Theorem 2.1. Taking into account all the properties of Φ, the Kakutani-Ky Fan fixed point Theorem [8], [14] can be applied.

Indeed, $\Phi:\mathcal{Z}_M\to 2^{\mathcal{Z}_M}$ is defined on $\mathcal{Z}_M$ which is a convex, compact and non-empty set; it is also upper-semicontinuous and its set-values $\Phi(Z)$ are convex, closed and non-empty in $\mathcal{Z}_M$. From Kakutani-Ky Fan fixed point Theorem it follows that the multifunction Φ has at least a fixed point, i.e. there exists at least an element $Z\in\mathcal{Z}_M$ such that $Z\in\Phi(Z)$, hence $Z=U$; but U is of the form (3.10), therefore this fixed point Z is a solution of Cauchy-Ionescu Problem (1.1) + (1.2).

(ii) We denote by S_α the set of solutions to Problem (1.1)+(1.2), a notation showing that any solution Z depends on the function α defined by (3.1). The set S_α contains at least an element. The set S_α is *compact, non-empty* in the Banach space $C(D_0;\mathbb{R}^n)$, being the set of the fixed points of multifunction Φ.

(*iii*) The graph $\mathcal{H}$ of the multifunction $\alpha \to S_\alpha$, defined on $C^*(D_0;\mathbb{R}^n)$ with values in $2^{\mathcal{Z}_M}$, $S_\alpha \subset \Phi(\mathcal{Z}_M) \subset 2^{\mathcal{Z}_M}$, is closed in $C^*(D_0;\mathbb{R}^n) \times \mathcal{Z}_M$ since $\mathcal{H}$ is the image of the compact set $\mathcal{H}_1$ of the triples $(\alpha, Z, U) \in \mathcal{G}$ with $Z = U$ through the projection mapping $(\alpha, Z, U) \to (\alpha, Z)$. The mapping $\alpha \to S_\alpha$ is — in general — a multifunction because several solutions of the Problem (1.1)+(1.2) can exist, which are fixed points of mapping Φ corresponding to the same function α. Because the mapping $\alpha \to S_\alpha$ has a closed graph $\mathcal{H}$ by Theorem 2.1, it follows that $\alpha \to S_\alpha$ is upper-semicontinuous on $C^*(D_0;\mathbb{R}^n)$, what completes the proof.

Remarks.

a) The proof is similar to the case $y \geq \varphi(x),\ \ 0 \leq x \leq a$.

b) We have not considered the case when the curve γ is tangent to one of the "characteristics" $x = 0$ or $y = 0$ at one of its points, at least.

c) If γ is a curve segment of a "characteristic", Cauchy-Ionescu's problem is impossible or non-determinate.

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