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# ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO A NONLINEAR HYPERBOLIC PROBLEM

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**Abstract.** We study the asymptotic behaviour of the solutions to a nonlinear hyperbolic system which occurs in integrated circuits modelling.

**Keywords:** hyperbolic system, maximal monotone operator, Cauchy problem, $\omega$-limit sets.

**AMS Subject Classification:** 35L50, 47H05.

## 1 Introduction

We consider the following nonlinear hyperbolic system

$$\text{(S)} \qquad \begin{cases} \dfrac{\partial i_k}{\partial t} + \dfrac{\partial v_k}{\partial x} + \alpha_k(x, i_k) = f_k(t,x) \\ \\ \dfrac{\partial v_k}{\partial t} + \dfrac{\partial i_k}{\partial x} + \beta_k(x, v_k) = g_k(t,x), \end{cases}$$

$$t > 0, \;\; 0 < x < 1, \;\; k = \overline{1,n}$$

with the boundary condition

$$\text{(BC)} \qquad \begin{pmatrix} col\,(i_1(t,0), -i_1(t,1), \ldots, i_n(t,0), -i_n(t,1)) \\ S\,(col\,(w_1'(t), \ldots, w_m'(t))) \end{pmatrix} \in$$

$$\in -G \begin{pmatrix} col\,(v_1(t,0), v_1(t,1), \ldots, v_n(t,0), v_n(t,1)) \\ col\,(w_1(t), \ldots, w_m(t)) \end{pmatrix}, \quad t > 0$$

and the initial data

$$\text{(IC)} \qquad \begin{cases} i_k(0,x) = i_{k0}(x), \;\; v_k(0,x) = v_{k0}(x), \quad 0 < x < 1, \;\; k = \overline{1,n}, \\ w_j(0) = w_{j0}, \;\; j = \overline{1,m}. \end{cases}$$

This problem has applications in the theory of integrated circuits and hydralics (see [2,3]). The existence, uniqueness and some regularity properties of the strong and weak solutions to the problem (S), (BC), (IC) have been investigated in [2]. In this paper we shall study the asymptotic behaviour of the solutions under some assumptions on the functions $\alpha_k$, $\beta_k$, $k = \overline{1,n}$ and on the operator $G$. For the basic concepts and results in the theory of monotone operators and nonlinear evolution equations of monotone type we refer the reader to [1,4].

## 2 Results

We introduce the assumptions that we shall use in this paper

$$
\text{(A1)}\qquad \left\{
\begin{array}{l}
\text{(a) The functions } x\longrightarrow\alpha_k(x,p) \text{ and } x\longrightarrow\beta_k(x,p) \text{ are in } L^2(0,1) \text{ for any}\\
\quad\text{fixed } p\in\mathbb{R}. \text{ Besides the functions } p\longrightarrow\alpha_k(x,p) \text{ and } p\longrightarrow\beta_k(x,p)\\
\quad\text{are continuous from } \mathbb{R} \text{ to } \mathbb{R}, \text{ for a.a. } x\in(0,1),\ k\in\overline{1,n}.\\
\text{(b) The functions } p\longrightarrow\alpha_k(x,p) \text{ and } p\longrightarrow\beta_k(x,p) \text{ are nondecreasing from}\\
\quad\mathbb{R} \text{ to } \mathbb{R}, \text{ for a.a. } x\in(0,1),\ \ k=\overline{1,n}.\\
\text{(c) The functions } p\longrightarrow\alpha_k(x,p) \text{ and } p\longrightarrow\beta_k(x,p) \text{ are (strictly) increasing}\\
\qquad\quad(\alpha_k(x,p_1)-\alpha_k(x,p_2))(p_1-p_2)>0,\\
\qquad\quad(\beta_k(x,p_1)-\beta_k(x,p_2))(p_1-p_2)>0,\\
\quad\text{for a.a. } x\in(0,1),\ (\forall)\,p_1,p_2\in\mathbb{R},\ p_1\neq p_2,\ k\in\overline{1,n}.\\
\text{(d) There exist } a_1,a_2>0 \text{ such that}\\
\qquad\quad(\alpha_k(x,p_1)-\alpha_k(x,p_2))(p_1-p_2)\geq a_1(p_1-p_2)^2,\\
\qquad\quad(\beta_k(x,p_1)-\beta_k(x,p_2))(p_1-p_2)\geq a_2(p_1-p_2)^2,\\
\quad\text{for a.a. } x\in(0,1),\ (\forall)\,p_1,p_2\in\mathbb{R},\ k\in\overline{1,n}.
\end{array}
\right.
$$

$$
\text{(A2)}\qquad \left\{
\begin{array}{l}
\text{(a) } G: D(G)\subset\mathbb{R}^{2n+m}\longrightarrow\mathbb{R}^{2n+m} \text{ is a maximal monotone mapping}\\
\quad\text{(possibly multivalued), } D(G)\neq\emptyset. \text{ Moreover, } G=\begin{pmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix} \text{ with}\\
\qquad\quad G_{11}: D(G_{11})\subset\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n},\quad G_{12}: D(G_{12})\subset\mathbb{R}^{m}\longrightarrow\mathbb{R}^{2n},\\
\qquad\quad G_{21}: D(G_{21})\subset\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{m},\quad G_{22}: D(G_{22})\subset\mathbb{R}^{m}\longrightarrow\mathbb{R}^{m}.\\
\text{(b) If } x,y\in D(G),\ x=col(x^a,x^b)\in\mathbb{R}^{2n}\times\mathbb{R}^m,\ y=col(y^a,y^b)\in\\
\quad\in\mathbb{R}^{2n}\times\mathbb{R}^m, \text{ with } x^b\neq y^b \text{ and } w_1\in G(x),\ w_2\in G(y), \text{ then}\\
\qquad\quad < w_1-w_2, x-y>_{\mathbb{R}^{2n+m}}>0.\\
\text{(c) There exists } K>0 \text{ such that } (\forall)\,x,y\in D(G),\ x=col(x^a,x^b),\\
\quad y=col(y^a,y^b) \text{ and } (\forall)\,w_1\in G(x),\ w_2\in G(y) \text{ we have}\\
\qquad\quad < w_1-w_2, x-y>_{\mathbb{R}^{2n+m}}\geq K\|x^b-y^b\|^2_{\mathbb{R}^m}.
\end{array}
\right.
$$

$$
\text{(A3)}\qquad S=diag(s_1,\ldots,s_m)\quad \text{with}\quad s_j>0,\ j=\overline{1,m}.
$$

**Remark.** Our assumption (A2)a is a technical one and it is automatically satisfied if $G$ is a matrix.

First, we presente some existence and uniqueness results for the solutions of the problem (S), (BC), (IC) obtained in [2,3]. We consider the following *spaces* $X=(L^2(0,1))^{2n}$, $\mathbb{R}^m$ and $Y=X\times\mathbb{R}^m$ with the corresponding scalar products

$$
\begin{aligned}
&(f,g)_X=\sum_{k=1}^{2n}\int_0^1 f_k(x)g_k(x)\,dx,\quad f=col\,(f_1,\ldots,f_{2n}),\quad g=col\,(g_1,\ldots,g_{2n})\in X,\\
&\langle x,y\rangle_s=\sum_{i=1}^{m}s_ix_iy_i,\quad x,\ y\in\mathbb{R}^m,\\
&\left\langle\begin{pmatrix} f\\ x\end{pmatrix},\begin{pmatrix} g\\ y\end{pmatrix}\right\rangle_Y=(f,g)_X+\langle x,y\rangle_s,\quad \begin{pmatrix} f\\ x\end{pmatrix},\ \begin{pmatrix} g\\ y\end{pmatrix}\in Y.
\end{aligned}
$$

We define the *operator* $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$,

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y;\ i_k,\ v_k \in H^1(0,1),\ \ k = \overline{1,n},\ \ \begin{pmatrix} \gamma_0 v \\ w \end{pmatrix} \in D(G) \text{ and } \right.$$

$$\left. \gamma_1 i \in -G_{11}(\gamma_0 v) - G_{12}(w) \right\},$$

where $i = col\,(i_1, \ldots, i_n)$, $v = col\,(v_1, \ldots, v_n)$, $w = col\,(w_1, \ldots, w_m)$,
$\gamma_1 i = col\,(i_1(0), -i_1(1), \ldots, i_n(0), -i_n(1))$, $\gamma_0 v = col\,(v_1(0), v_1(1), \ldots, v_n(0), v_n(1))$,

$$\mathcal{A}\begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} col\,(v_1', v_2', \ldots, v_n') \\ col\,(i_1', i_2', \ldots, i_n') \\ S^{-1}G_{21}(\gamma_0 v) + S^{-1}G_{22}(w) \end{pmatrix}.$$

We also define the *operator* $\mathcal{B} : D(\mathcal{B}) \subset Y \to Y$

$$\mathcal{B}\begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} col\,(\alpha_1(\cdot, i_1), \alpha_2(\cdot, i_2), \ldots, \alpha_n(\cdot, i_n)) \\ col\,(\beta_1(\cdot, v_1), \beta_2(\cdot, v_2), \ldots, \beta_n(\cdot, v_n)) \\ 0 \end{pmatrix},$$

$$D(\mathcal{B}) = \{u \in Y,\ \ u = col\,(i, v, w);\ \ \mathcal{B}(u) \in Y\}.$$

**Remark.** Under the assumptions (A1)ab, (A2)a, (A3) we can show that $\overline{D(\mathcal{A})} = X \times \overline{D(G_{12}) \cap D(G_{22})}^{\mathbb{R}^m}$ and $D(\mathcal{A}) \subset D(\mathcal{B})$, (see [3]).

**Lemma 2.1.** *If (A1)ab, (A2)a and (A3) hold, then the operator $\mathcal{A}+\mathcal{B}$ is maximal monotone.*

For the proof of Lemma 2.1 see [2].

Using the operators $\mathcal{A}$ and $\mathcal{B}$ our problem (S), (BC), (IC) can be equivalently expressed as the following Cauchy problem in the space $Y$

$$\text{(P)} \qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} i \\ v \\ w \end{pmatrix} + \mathcal{A}\begin{pmatrix} i \\ v \\ w \end{pmatrix} + \mathcal{B}\begin{pmatrix} i \\ v \\ w \end{pmatrix} \ni \begin{pmatrix} f(t,\cdot) \\ g(t,\cdot) \\ 0 \end{pmatrix}, \quad t > 0 \\ \begin{pmatrix} i(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix}, \end{cases}$$

where $f = col\,(f_1, \ldots, f_n)$, $g = col\,(g_1, \ldots, g_n)$, $i^0 = col\,(i_{10}, \ldots, i_{n0})$, $v^0 = col\,(v_{10}, \ldots, v_{n0})$, $w^0 = col\,(w_{10}, \ldots, w_{m0})$.

We say that $u = col\,(i, v, w)$ is a strong (weak) solution for the problem (S), (BC), (IC) if $u$ is a strong (respectively weak) solution for the problem $(\mathrm{P})_1$ (see [1, Ch.III]).

**Theorem 2.1.** *a) Assume that (A1)ab, (A2)a and (A3) hold. If $i_{k0},\ v_{k0} \in H^1(0,1)$, $k = \overline{1,n}$, $w^0 \in \mathbb{R}^m$, $\begin{pmatrix} \gamma_0 v^0 \\ w^0 \end{pmatrix} \in D(G)$, $\gamma_1 i^0 \in -G_{11}(\gamma_0 v^0) - G_{12}(w^0)$ and $f_k,\ g_k \in W^{1,1}(0,T; L^2(0,1))$, $k = \overline{1,n}$ (with $T > 0$ fixed), then the problem (S), (BC), (IC) has a unique strong solution $u = col\,(i, v, w) \in W^{1,\infty}(0,T; Y)$. Moreover*

$$i_k,\ v_k \in L^\infty(0,T; H^1(0,1)), \quad \textit{hence} \quad i_k,\ v_k \in L^\infty((0,T) \times (0,1)), \quad k = \overline{1,n}. \tag{1}$$

*b) Assume that (A1)ab, (A2)a and (A3) hold. If* $i_{k0}$, $v_{k0} \in L^2(0,1)$, $k=\overline{1,n}$, $w^0 \in \overline{D(G_{12}) \cap D(G_{22})}$ *and* $f_k, g_k \in L^1(0,T;L^2(0,1))$, $k=\overline{1,n}$ *(with* $T>0$ *fixed), then the problem (S), (BC), (IC) has a unique weak solution* $u = col\,(i,v,w) \in C([0,T];Y)$.

For the proof of Theorem 2.1 see [2].

**Theorem 2.2.** *Assume that (A1)ad, (A2)ac and (A4) hold. Then the equation (the stationary problem)*

$$\mathcal{A}(u) + \mathcal{B}(u) \ni 0 \tag{2}$$

*has a unique solution* $u = col\,(i,v,w) \in D(\mathcal{A})$

**Lemma 2.2.** *If (A1)ab, (A2)a and (A3) hold, then for every* $\lambda > 0$ *the operator* $(I + \lambda(\mathcal{A}+\mathcal{B}))^{-1}$ *is compact in the space* $Y$.

The first asymptotic result for the solutions of the problem (S), (BC), (IC) is

**Theorem 2.3.** *Assume that (A1)ac, (A2)ab and (A3) hold and* $i_{k0}$, $v_{k0} \in L^2(0,1)$, $k=\overline{1,n}$, $w^0 \in \overline{D(G_{12}) \cap D(G_{22})}$, $f_k, g_k \in L^1(\mathbb{R}_+;L^2(0,1))$, $k=\overline{1,n}$. *In addition, suppose that there exists a solution to the stationary problem (2). Then, this solution is unique, denoted by* $\gamma = col\,(p,q,r)$ *and the weak solution* $\{u(t) = col\,(i(t),v(t),w(t)),\ t \geq 0\}$ *of the problem (S), (BC), (IC) satisfies*

$$u(t) \to \gamma, \ \ as \ \ t \to \infty, \ \ in \ Y. \tag{3}$$

*If* $col(i^0,v^0,w^0) \in D(\mathcal{A})$, $f_k, g_k \in W^{1,1}(\mathbb{R}_+;L^2(0,1))$, $k=\overline{1,n}$, *then the strong solution* $\{u(t) = col\,(i(t),v(t),w(t)),\ t \geq 0\}$ *of the problem (S), (BC), (IC) also satisfies*

$$\begin{gathered} i_k(t,\cdot) \to p_k, \ \ v_k(t,\cdot) \to q_k, \ \ k=\overline{1,n}, \\ for \ \ t \to \infty, \ \ strongly \ \ in \ \ C([0,1]), \end{gathered} \tag{4}$$

*(eventually on some subsequences).*

**Remark.** Suppose the assumptions of Theorem 2.2 hold and $u^0 = col\,(i^0,v^0,w^0) \in D(\mathcal{A})$. Let $\gamma = col\,(p,q,r)$ be the unique solution of the equation (2) and $u = col\,(i,v,w)$ the strong solution of the problem (S), (BC), (IC) with $f = g \equiv 0$. Then, using {[4], Theor.2.6, p.116} we have

$$\|u(t) - \gamma\|_Y \leq e^{-C_0 t} \left\|u^0 - \gamma\right\|_Y, \ \ t \geq 0,$$

(see the proof of Theorem 2.2).

**Remark.** Under the assumptions of Theorem 2.3, using (3) we deduce that the average $\sigma_u$ of the solution $u = col\,(i,v,w)$, that is

$$\sigma_u(t) = \frac{1}{t}\int_0^t u(s)\,ds$$

converges to $\gamma$, as $t \to \infty$, in $Y$.

To present the following asymptotic result for the solutions of the problem (S), (BC), (IC) we introduce the assumptions

$$\text{(A4)}\quad \begin{cases} \text{(a) The functions } p \to \alpha_k(x,p),\ \ k=\overline{1,n} \text{ are (strictly) increasing, for a.a.}\\ \qquad x \in (0,1).\\ \text{(b) } G \text{ satisfies the condition}\\ \qquad \text{If } G\,(col\,(x_1, x_2, \dots, x_{2n-1}, x_{2n}, z_{2n+1}, \dots, z_{2n+m})) \cap\\ \qquad\quad \cap\, G\,(col\,(y_1, y_2, \dots, y_{2n-1}, y_{2n}, z_{2n+1}, \dots, z_{2n+m})) \neq \emptyset \qquad (5)\\ \qquad \text{then } x_{2i} = y_{2i},\ \ i=\overline{1,n} \ \text{ or } \ x_{2i-1} = y_{2i-1},\ \ i=\overline{1,n}. \end{cases}$$

$$\text{(A5)}\quad \begin{cases} \text{(a) The functions } p \to \beta_k(x,p),\ \ k=\overline{1,n} \text{ are (strictly) increasing, for a.a.}\\ \qquad x \in (0,1).\\ \text{(b) } G^{-1} \text{ satisfies (5).} \end{cases}$$

**Theorem 2.4.** *Assume that (A1)ab, (A2)ab, (A3) and [(A4) or (A5)] hold and* $i_{k0}$, $v_{k0} \in L^2(0,1)$, $k=\overline{1,n}$, $w^0 \in \overline{D(G_{12})} \cap \overline{D(G_{22})}$, $f_k,\ g_k \in L^1(\mathbb{R}_+; L^2(0,1))$, $k=\overline{1,n}$. *In addition, suppose that there exists a solution of the equation (2). Then this solution is unique, denoted by* $\gamma = col\,(p,q,r)$ *and the weak solution* $\{u(t) = col\,(i(t), v(t), w(t)),\ t \geq 0\}$ *of the problem (S), (BC), (IC) satisfies*

$$u(t) \to \gamma, \ \ \text{as } t \to \infty, \ \ \text{in } Y.$$

*If* $col\,(i^0, v^0, w^0) \in D(\mathcal{A})$ *and* $f_k,\ g_k \in W^{1,1}(\mathbb{R}_+; L^2(0,1))$, $k=\overline{1,n}$, *then occurs also (4).*

**Remark.** If the assumption (5) is replaced by the following weakly condition

$$\left.\begin{array}{c} \text{If } \ G\,(col\,(x_1, x_2, \dots, x_{2n-1}, x_{2n}, z_{2n+1}, \dots, z_{2n+m})) \cap \\ \cap\, G\,(col\,(y_1, y_2, \dots, y_{2n-1}, y_{2n}, z_{2n+1}, \dots, z_{2n+m})) \neq \emptyset \\ \text{and } \ x_{2i-1} - y_{2i-1} = x_{2i} - y_{2i}, \ \ i=\overline{1,n} \\ \text{then } \ x_{2i} = y_{2i}, \ \ i=\overline{1,n} \ \text{ or } \ x_{2i-1} = y_{2i-1}, \ \ i=\overline{1,n} \end{array}\right] \tag{6}$$

the conclusions of the Theorem 4 remain also true.

## 3 Proofs

**Proof of Theorem 2.2.** We suppose without loss of generality that $G$ is single-valued. Under the assumptions of the theorem, the operator $\mathcal{A}+\mathcal{B}$ is strongly monotone. Indeed, we have

$$\begin{aligned} &\langle (\mathcal{A}+\mathcal{B})(u) - (\mathcal{A}+\mathcal{B})(\widetilde{u}), u - \widetilde{u} \rangle_Y = \\ &= \langle G\begin{pmatrix} \gamma_0 v \\ w \end{pmatrix} - G\begin{pmatrix} \gamma_0 \widetilde{v} \\ \widetilde{w} \end{pmatrix}, \begin{pmatrix} \gamma_0 (v - \widetilde{v}) \\ w - \widetilde{w} \end{pmatrix} \rangle_{\mathbb{R}^{2n+m}} + \\ &+ \sum_{k=1}^{n} \int_0^1 [\alpha_k(x, i_k(x)) - \alpha_k(x, \widetilde{i}_k(x))] \cdot [i_k(x) - \widetilde{i}_k(x)]\, dx + \\ &+ \sum_{k=1}^{n} \int_0^1 [\beta_k(x, v_k(x)) - \beta_k(x, \widetilde{v}_k(x))] \cdot [v_k(x) - \widetilde{v}_k(x)]\, dx \geq \end{aligned}$$

$$\geq K\|w-\widetilde{w}\|^2_{\mathbb{R}^m} + a_1\sum_{k=1}^{n}\int_0^1 |i_k(x)-\widetilde{i}_k(x)|^2\,dx + a_2\sum_{k=1}^{n}\int_0^1 |v_k(x)-\widetilde{v}_k(x)|^2\,dx \geq$$

$$\geq C_0\,\|u-\widetilde{u}\|^2_Y\,,\quad (C_0>0),\quad (\forall)\,u=col\,(i,v,w),\ \ \widetilde{u}=col\,(\widetilde{i},\widetilde{v},\widetilde{w})\in D(\mathcal{A}).$$

So, the operator $\mathcal{A}+\mathcal{B}$ is coercive and then $R(\mathcal{A}+\mathcal{B})=Y$. Therefore, we deduce that the equation (2) has a unique solution $u=col\,(i,v,w)\in D(\mathcal{A})$. □

**Proof of Lemma 2.2.** Suppose again that $G$ is single-valusd. Let be $\lambda>0$ and $M=\{col\,(a^i,b^i,c^i);\ \ i\in I\}$ a bounded set in $Y$. We shall prove that $(I+\lambda(\mathcal{A}+\mathcal{B}))^{-1}(M)$ is a bounded set in the space $(H^1(0,1))^{2n}\times\mathbb{R}^m$. For, because the operator $(I+\lambda(\mathcal{A}+\mathcal{B}))^{-1}$ is nonexpansive in $Y$, we firstly deduce that the set $\{col\,(p^i,q^i,r^i);\ \ i\in I\}$ is bounded in $Y$, where

$$\begin{pmatrix}p^i\\ q^i\\ r^i\end{pmatrix}=(I+\lambda(\mathcal{A}+\mathcal{B}))^{-1}\begin{pmatrix}a^i\\ b^i\\ c^i\end{pmatrix}\ \Leftrightarrow\ \begin{pmatrix}a^i\\ b^i\\ c^i\end{pmatrix}=\begin{pmatrix}p^i\\ q^i\\ r^i\end{pmatrix}+\lambda(\mathcal{A}+\mathcal{B})\begin{pmatrix}p^i\\ q^i\\ r^i\end{pmatrix},\ \ i\in I$$

The last relation is equivalent to the system

$$\begin{cases} a^i_k=p^i_k+\lambda[(q^i_k)'+\alpha_k(x,p^i_k(x))]\\ b^i_k=q^i_k+\lambda[(p^i_k)'+\beta_k(x,q^i_k(x))],\quad k=\overline{1,n},\ \ 0<x<1\\ c^i_j=r^i_j+\lambda[S^{-1}G_{21}(\gamma_0q^i)+S^{-1}G_{22}(r^i)]_j,\ \ j=\overline{1,m}\\ \gamma_1p^i=-G_{11}(\gamma_0q^i)-G_{12}(r^i)\end{cases}$$

or

$$\begin{cases} \dfrac{1}{\lambda}p^i_k+(q^i_k)'+\alpha_k(x,p^i_k(x))=\dfrac{1}{\lambda}a^i_k\\ \dfrac{1}{\lambda}q^i_k+(p^i_k)'+\beta_k(x,q^i_k(x))=\dfrac{1}{\lambda}b^i_k,\ \ k=\overline{1,n},\ \ 0<x<1\\ \dfrac{1}{\lambda}r^i_j+[S^{-1}G_{21}(\gamma_0q^i)+S^{-1}G_{22}(r^i)]_j=\dfrac{1}{\lambda}c^i_j,\ \ j=\overline{1,m}\\ \gamma_1p^i=-G_{11}(\gamma_0q^i)-G_{12}(r^i).\end{cases}\tag{7}$$

In what follows we shall prove that the sets $\{(p^i_k)';\ \ i\in I\}$ and $\{(q^i_k)';\ \ i\in I\}$ are bounded in $L^1(0,1)$, $\ k=\overline{1,n}$. To this purpose we define the functions

$$f^i_k(x)=\begin{cases}(q^i_k)'(x)/|(q^i_k)'(x)|, & \text{if }\ (q^i_k)'(x)\neq 0,\\ 0, & \text{if }\ (q^i_k)'(x)=0,\end{cases}$$
$$g^i_k(x)=\begin{cases}(p^i_k)'(x)/|(p^i_k)'(x)|, & \text{if }\ (p^i_k)'(x)\neq 0,\\ 0, & \text{if }\ (p^i_k)'(x)=0,\end{cases}$$
$$0<x<1,\ \ k=\overline{1,n},\ \ i\in I.$$

We multiply the equations $(7)_{1,2}$ by $f^i_k(x)+p^i_k(x)-i^0_k(x)$ and $g^i_k(x)+q^i_k(x)-v^0_k(x)$ respectively $(col\,(i^0,v^0,w^0)\in D(\mathcal{A}))$. Then integrating over $[0,1]$, we obtain, after some computations, that

$$\int_0^1|(q^i_k)'(x)|\,dx\leq-\int_0^1[p^i_k(x)-i^0_k(x)]\cdot[q^i_k(x)-v^0_k(x)]'\,dx+const.,\ \ k=\overline{1,n},\ \ (\forall)\,i\in I\tag{8}$$

and

$$\int_0^1|(p^i_k)'(x)|\,dx\leq-\int_0^1[p^i_k(x)-i^0_k(x)]'\cdot[q^i_k(x)-v^0_k(x)]\,dx+const.,\ \ k=\overline{1,n},\ \ (\forall)\,i\in I,\tag{9}$$

(*const.* is a positive constant).

Next, we substract the relations

$$\frac{1}{\lambda}r_j^i + [S^{-1}G_{21}(\gamma_0 q^i) + S^{-1}G_{22}(r^i)]_j = \frac{1}{\lambda}c_j^i,$$

$$\frac{1}{\lambda}w_j^0 + [S^{-1}G_{21}(\gamma_0 v^0) + S^{-1}G_{22}(w^0)]_j =: \frac{1}{\lambda}\tilde{\delta}_j^0, \ \ (j = \overline{1,m}),$$

we multiply the obtained relation by $s_j(r_j^i - w_j^0)$ and then by adding $(j = \overline{1,m})$, we obtain

$$\frac{1}{\lambda}\|r^i - w^0\|_s^2 + \langle G_{21}(\gamma_0 q^i) + G_{22}(r^i) - G_{21}(\gamma_0 v^0) - G_{22}(w^0), r^i - w^0\rangle_{\mathbb{R}^m} \leq const., \ \ (\forall)\, i \in I. \tag{10}$$

By (8), (9) and (10) we deduce

$$\sum_{k=1}^{n}\left[\int_0^1 (|(q_k^i)'| + |(p_k^i)'|)\, dx\right] + \frac{1}{\lambda}\|r^i - w^0\|_s^2 \leq const., \ \ (\forall)\, i \in I$$

Therefore, the sets $\{(p_k^i)';\ i \in I\}$ and $\{(q_k^i)';\ i \in I\}$, $k = \overline{1,n}$ are bounded in $L^1(0,1)$. Because the sets $\{p_k^i;\ i \in I\}$ and $\{q_k^i;\ i \in I\}$, $k = \overline{1,n}$ are bounded in $L^2(0,1)$, if follows that $\{p_k^i;\ i \in I\}$ and $\{q_k^i;\ i \in I\}$, $k = \overline{1,n}$ are bounded in $C([0,1])$. Using (A1)ab we deduce that the sets $\{\alpha_k(\cdot, p_k^i);\ i \in I\}$ and $\{\beta_k(\cdot, q_k^i);\ i \in I\}$, $k = \overline{1,n}$ are bounded in $L^2(0,1)$. Therefore, by $(7)_{1,2}$ we obtain that the sets $\{(p_k^i)';\ i \in I\}$ and $\{(q_k^i)';\ i \in I\}$, $k = \overline{1,n}$ are bounded in $L^2(0,1)$. Hence the set $\{col\,(p^i, q^i, r^i);\ i \in I\}$ is bounded in $(H^1(0,1))^{2n} \times \mathbb{R}^m$ and so the operator $(I + \lambda(\mathcal{A} + \mathcal{B}))^{-1}$ is compact in $Y$. □

**Proof of Theorem 2.3.** Assume that $G$ is single-valued. Using {[4], Theor.1.3, p.77} and {[4], Propo.2.4, p.107} it is sufficient to prove the theorem for $f = g \equiv 0$ and $u^0 = col\,(i^0, v^0, w^0) \in D(\mathcal{A})$. Then

$$u(t) = col\,(i(t), v(t), w(t)) = S(t)u^0, \ \ t \geq 0,$$

where $\{S(t);\ t \geq 0\}$ is the semigroup generated by $-(\mathcal{A} + \mathcal{B})$. Because the operator $(I + \lambda(\mathcal{A}+\mathcal{B}))^{-1}$ is compact in the space $Y$ (Lemma 2.2) and $(\mathcal{A}+\mathcal{B})^{-1}(0) \neq 0$, by {[4], Propo.2.5, p.108} we deduce that the set $\{S(t)u^0;\ t \geq 0\}$ is relatively compact in $Y$.

Using the assumptions of the theorem, we obtain that the operator $\mathcal{A} + \mathcal{B}$ is strictly monotone. So, the stationary problem (2) has a unique solution, denoted by $\gamma =$ $= col\,(p, q, r)$. Using {[4], Corollary 2.2, p.126} we conclude that

$$S(t)u^0 \to \gamma, \ \text{ as } t \to \infty, \ \text{ in } Y.$$

In what follows we suppose that $u^0 = col\,(i^0, v^0, w^0) \in D(\mathcal{A})$ and $f_k,\ g_k \in W^{1,1}(\mathbb{R}_+; L^2(0,1))$, $k = \overline{1,n}$. We shall prove that the sets

$$\{i_k(t,\cdot);\ t \geq 0\} \ \text{ and } \ \{v_k(t,\cdot);\ t \geq 0\} \ \text{ are bounded in } H^1(0,1), \ k = \overline{1,n}. \tag{11}$$

In this case $\{u(t,\cdot) = col\,(i(t,\cdot), v(t,\cdot), w(t));\ \ t \geq 0\}$ is the strong solution of the problem (S), (BC), (IC) and

$$\left\|\frac{d^+u}{dt}(t,\cdot)\right\|_Y \leq \left\|(\mathcal{A}+\mathcal{B})(u^0) - F(0,\cdot)\right\|_Y + \int_0^t \left\|\frac{dF}{ds}(s,\cdot)\right\|_Y ds,$$

where $F(t,\cdot) = col\,(f(t,\cdot), g(t,\cdot), 0)$.

Therefore the set

$$\left\{\frac{d^+u}{dt}(t,\cdot),\ \ t > 0\right\} \text{ is bounded in } Y. \tag{12}$$

Using now the inequality

$$\|u(t,\cdot) - \gamma\|_Y \leq \left\|u^0 - \gamma\right\|_Y + \int_0^t \|F(s,\cdot)\|_Y\ ds,\ \ t \geq 0$$

we deduce that the set

$$\{u(t,\cdot);\ \ t \geq 0\} \text{ is bounded in } Y. \tag{13}$$

By the equation $(\mathrm{P})_1$ we obtain

$$u(t,\cdot) = (I + (\mathcal{A}+\mathcal{B}))^{-1}\left(u(t,\cdot) - \frac{d^+u}{dt}(t,\cdot) + F(t,\cdot)\right),\ \ t \geq 0. \tag{14}$$

Because $f_k, g_k \in W^{1,1}(\mathbb{R}_+; L^2(0,1))$, $k = \overline{1,n}$, by the relations (12), (13), (14) and Lemma 2.2, we deduce (11). So, by (3) and (11) we obtain

$i_k(t,\cdot) \to p_k$, weakly in $H^1(0,1)$,
$v_k(t,\cdot) \to q_k$, weakly in $H^1(0,1)$,
$i_k(t,x) \to p_k(x)$, uniformly in $x \in [0,1]$,
$v_k(t,x) \to q_k(x)$, uniformly in $x \in [0,1]$, $k = \overline{1,n}$,
$w_j(t) \to r_j$, $j = \overline{1,m}$,

as $t \to \infty$ (eventually on some subsequences). □

**Proof of Theorem 2.4.** Suppose again that $G$ is single-valued. As in the proof of Theorem 2.3, it is sufficient to prove the theorem in the case $f = g \equiv 0$ and $u^0 = col(i^0, v^0, w^0) \in D(\mathcal{A})$. Using Lemma 2.2 and {[4], Propo.2.5, p.108} we also deduce that the orbit throught $u^0$ of the semigroup $S(t)$, $\gamma(u^0) = \{S(t)(u^0);\ \ t \geq 0\}$ is relatively compact in the space $Y$. Denote by $F$ the set of the equilibrium points of the semigroup $S(t)$, that is $F = (\mathcal{A}+\mathcal{B})^{-1}(0)\,(\neq \emptyset)$. We shall prove that $F$ consists of a single element. For this purpose, let be $\gamma = col\,(p,q,r)$, $\bar{\gamma} = col\,(\bar{p}, \bar{q}, \bar{r}) \in F$. Then we have

$$\langle(\mathcal{A}+\mathcal{B})(\gamma) - (\mathcal{A}+\mathcal{B})(\bar{\gamma}), \gamma - \bar{\gamma}\rangle_Y = 0.$$

By the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, the above relation gives us

$$\left\langle \begin{pmatrix} col\,(q_1',\dots,q_n') + col\,(\alpha_1(\cdot,p_1),\dots,\alpha_n(\cdot,p_n)) \\ col\,(p_1',\dots,p_n') + col\,(\beta_1(\cdot,q_1),\dots,\beta_n(\cdot,q_n)) \\ S^{-1}G_{21}(\gamma_0 q) + S^{-1}G_{22}(r) \end{pmatrix} - \right.$$
$$\left. - \begin{pmatrix} col\,(\bar{q}_1',\dots,\bar{q}_n') + col\,(\alpha_1(\cdot,\bar{p}_1),\dots,\alpha_n(\cdot,\bar{p}_n)) \\ col\,(\bar{p}_1',\dots,\bar{p}_n') + col\,(\beta_1(\cdot,\bar{q}_1),\dots,\beta_n(\cdot,\bar{q}_n)) \\ S^{-1}G_{21}(\gamma_0 \bar{q}) + S^{-1}G_{22}(\bar{r}) \end{pmatrix}, \begin{pmatrix} col\,(p_1-\bar{p}_1,\dots,p_n-\bar{p}_n) \\ col\,(q_1-\bar{q}_1,\dots,q_n-\bar{q}_n) \\ col\,(r_1-\bar{r}_1,\dots,r_m-\bar{r}_m) \end{pmatrix} \right\rangle_Y = 0, \tag{15}$$

$$\text{with } \gamma_1 p = -G_{11}(\gamma_0 q) - G_{12}(r), \ \ \gamma_1 \bar{p} = -G_{11}(\gamma_0 \bar{q}) - G_{12}(\bar{r}). \tag{16}$$

The relation (15) is equivalent to

$$\sum_{k=1}^{n} \int_0^1 (q_k-\bar{q}_k)'(p_k-\bar{p}_k)\,dx + \sum_{k=1}^{n} \int_0^1 (q_k-\bar{q}_k)(p_k-\bar{p}_k)'\,dx +$$
$$+ \langle S^{-1}G_{21}(\gamma_0 q) + S^{-1}G_{22}(r) - S^{-1}G_{21}(\gamma_0 \bar{q}) - S^{-1}G_{22}(\bar{r}), r-\bar{r}\rangle_s +$$
$$+ \sum_{k=1}^{n} \int_0^1 [\alpha_k(x,p_k(x)) - \alpha_k(x,\bar{p}_k(x))] \cdot [p_k(x)-\bar{p}_k(x)]\,dx +$$
$$+ \sum_{k=1}^{n} \int_0^1 [\beta_k(x,q_k(x)) - \beta_k(x,\bar{q}_k(x))] \cdot [q_k(x)-\bar{q}_k(x)]\,dx = 0$$

or

$$\begin{gathered} \langle G\begin{pmatrix}\gamma_0 q \\ r\end{pmatrix} - G\begin{pmatrix}\gamma_0 \bar{q} \\ \bar{r}\end{pmatrix}, \begin{pmatrix}\gamma_0(q-\bar{q}) \\ r-\bar{r}\end{pmatrix}\rangle_{\mathbb{R}^{2n+m}} + \\ + \sum_{k=1}^{n} \int_0^1 [\alpha_k(x,p_k(x)) - \alpha_k(x,\bar{p}_k(x))] \cdot [p_k(x)-\bar{p}_k(x)]\,dx + \\ + \sum_{k=1}^{n} \int_0^1 [\beta_k(x,q_k(x)) - \beta_k(x,\bar{q}_k(x))] \cdot [q_k(x)-\bar{q}_k(x)]\,dx = 0 \end{gathered} \tag{17}$$

with $p_k,\ \bar{p}_k,\ q_k,\ \bar{q}_k \in H^1(0,1);\ \ r,\ \bar{r} \in \mathbb{R}^m,\ \begin{pmatrix}\gamma_0 q \\ r\end{pmatrix},\ \begin{pmatrix}\gamma_0 \bar{q} \\ \bar{r}\end{pmatrix} \in D(G)$ and (16).

Using the monotonicity of the operator $G$ and of the functions $\alpha_k$ and $\beta_k$, we obtain by (17)

$$(\alpha_k(x,p_k(x)) - \alpha_k(x,\bar{p}_k(x)))(p_k(x)-\bar{p}_k(x)) = 0, \text{ for a.a. } x \in (0,1), \tag{18}$$

$$(\beta_k(x,q_k(x)) - \beta_k(x,\bar{q}_k(x)))(q_k(x)-\bar{q}_k(x)) = 0, \text{ for a.a. } x \in (0,1) \tag{19}$$

and by (A2)b if follows that $r = \bar{r}$.

Suppose firstly that (A4) hold. Then, by (18) we deduce that $p_k = \bar{p}_k,\ k = \overline{1,n}$ and using the relations

$$\begin{cases} q_k' + \alpha_k(\cdot,p_k) = 0 \\ p_k' + \beta_k(\cdot,q_k) = 0, \ \ k = \overline{1,n} \\ S^{-1}G_{21}(\gamma_0 q) + S^{-1}G_{22}(r) = 0 \\ \gamma_1 p = -G_{11}(\gamma_0 q) - G_{12}(r) \end{cases} \text{ and } \begin{cases} \bar{q}_k' + \alpha_k(\cdot,\bar{p}_k) = 0 \\ \bar{p}_k' + \beta_k(\cdot,\bar{q}_k) = 0, \ \ k = \overline{1,n} \\ S^{-1}G_{21}(\gamma_0 \bar{q}) + S^{-1}G_{22}(\bar{r}) = 0 \\ \gamma_1 \bar{p} = -G_{11}(\gamma_0 \bar{q}) - G_{12}(\bar{r}) \end{cases} \tag{20}$$

we obtain

$$q_k' = \bar{q}_k', \ \ k = \overline{1,n} \tag{21}$$

$$\text{and} \quad G\begin{pmatrix}\gamma_0 q\\ r\end{pmatrix} = G\begin{pmatrix}\gamma_0 \bar{q}\\ r\end{pmatrix}. \tag{22}$$

Using the condition (5) and (22) we deduce

$$q_k(0) = \bar{q}_k(0), \ \ k = \overline{1,n} \tag{23}$$

$$\text{or} \quad q_k(1) = \bar{q}_k(1), \ \ k = \overline{1,n}. \tag{24}$$

By {(21) and (23)} or {(21) and (24)} it follows that $q = \bar{q}$. So $col\,(p,q,r) = col\,(\bar{p},\bar{q},\bar{r})$, that is $F$ is a singleton.

Now suppose (A5) hold. By (19) we deduce that $q_k = \bar{q}_k, \ \ k = \overline{1,n}$ and by the relations (20) we obtain

$$p_k' = \bar{p}_k', \ \ k = \overline{1,n} \tag{25}$$

$$\text{and} \quad G^{-1}\begin{pmatrix}-\gamma_1 p\\ 0\end{pmatrix} \cap G^{-1}\begin{pmatrix}-\gamma_1 \bar{p}\\ 0\end{pmatrix} \neq \emptyset \tag{26}$$

Using the condition (5) for the operator $G^{-1}$ and (26) we deduce

$$p_k(0) = \bar{p}_k(0), \ \ k = \overline{1,n} \tag{27}$$

$$\text{or} \quad p_k(1) = \bar{p}_k(1), \ \ k = \overline{1,n}. \tag{28}$$

By {(25) and (27)} or {(25) and (28)} if follows that $p_k = \bar{p}_k, \ \ k = \overline{1,n}$. So, $col(p,q,r) = col\,(\bar{p},\bar{q},\bar{r})$, that is $F$ is a singleton.

Therefore we proved that under the assumptions (A4) or (A5), $F = \{\gamma\}$, $\gamma = col\,(p,q,r)$.

In what follows, let us denote by

$$\omega(u^0) = \{\sigma \in Y;\ (\exists)\, t_m \to \infty \text{ such that } S(t_m)u^0 \to \sigma, \text{ as } m \to \infty, \text{ strongly in } Y\}.$$

By {[4], Theor.2.8, p.121} we deduce that the set $\omega(u^0) \subset D(\mathcal{A})$. Let be $\sigma = col\,(a,b,c) \in \omega(u^0)$ arbitrary, but fixed for the time being. Then $S(t)\sigma$ satisfies the equation

$$\frac{d^+}{dt} S(t)\sigma + (\mathcal{A}+\mathcal{B})(S(t)\sigma) = 0, \ \ t \in \mathbb{R}_+. \tag{29}$$

On the other hand, {[4], Theor.2.7, p.118} gives us

$$\|S(t)\sigma - \gamma\|_Y = \|\sigma - \gamma\|_Y = const., \ \ t \in \mathbb{R}_+. \tag{30}$$

We multiply the equation (29) by $S(t)\sigma - \gamma$ in the space $Y$ and using (30) we obtain

$$\langle(\mathcal{A}+\mathcal{B})(S(t)\sigma), S(t)\sigma - \gamma\rangle_Y = 0, \ \ t \in \mathbb{R}_+. \tag{31}$$

We denote by $\widetilde{u}(t,\cdot) = col\,(\widetilde{i}(t,\cdot), \widetilde{v}(t,\cdot), \widetilde{w}(t)) = S(t)\sigma, \ \ t \in \mathbb{R}_+$. Then the relation (31) is equivalent to

$$\langle(\mathcal{A}+\mathcal{B})(\widetilde{u}(t,\cdot)) - (\mathcal{A}+\mathcal{B})(\gamma), \widetilde{u}(t,\cdot) - \gamma\rangle_Y = 0, \ \ t \in \mathbb{R}_+. \tag{32}$$

By the same arguments used above, (32) gives us that $\widetilde{w}(t) = r,\ (\forall)\, t \in \mathbb{R}_+$, so, in particular, $c = r$ and for any $t \in \mathbb{R}_+$

$$(\alpha_k(x, \widetilde{i}_k(t,x)) - \alpha_k(x, p_k(x)))(\widetilde{i}_k(t,x) - p_k(x)) = 0, \text{ for a.a. } x \in (0,1), \tag{33}$$

$$(\beta_k(x, \widetilde{v}_k(t,x)) - \beta_k(x, q_k(x)))(\widetilde{v}_k(t,x) - q_k(x)) = 0, \text{ for a.a. } x \in (0,1). \tag{34}$$

Suppose the assumption (A4) hold. Then, by (33) we obtain $\widetilde{i}_k(t,\cdot) = p_k,\ (\forall)\, t \in \mathbb{R}_+$, $k = \overline{1,n}$; in particular $a_k = p_k,\ k = \overline{1,n}$, so $a = p$ and using (29) we deduce that for any $t \in \mathbb{R}_+$

$$\frac{\partial \widetilde{v}_k}{\partial x}(t,\cdot) = q_k',\ \ k = \overline{1,n} \tag{35}$$

$$\text{and}\quad G\begin{pmatrix} \gamma_0 \widetilde{v}(t,\cdot) \\ r \end{pmatrix} = G\begin{pmatrix} \gamma_0 q \\ r \end{pmatrix}. \tag{36}$$

Using the condition (5), by the above relations we deduce that $\widetilde{v}(t,\cdot) = q,\ (\forall)\, t \in \mathbb{R}_+$, in particular $b = q$. So, $col\,(a,b,c) = col\,(p,q,r) \Leftrightarrow \sigma = \gamma$. The case (A5) is treated in a similar manner. Therefore in the both situations we obtain that $\omega(u^0) = \{\gamma\}$, that is

$$u(t) \to \gamma, \text{ as } t \to \infty, \text{ in } Y.$$

The last part of the theorem results using a similar reasoning as used in the proof of Theorem 2.3. □

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