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METHOD OF ROTHE IN PARABOLIC EQUATIONS INVOLVING SUBDIFFERENTIAL

CORINA L. CHIRIAC

Abstract. Approximation of a parabolic equation by a system of elliptic equations was introduced by E. Rothe [5] in 1930. The introduced Rothe's method is also called the method of lines or the method of discretization in time. In this paper we apply method of Rothe to parabolic equation involving subdifferential . We prove that the approximate problem is solvable and its solutions converge to the solution of the continuous problem.

Keywords: Parabolic equation, subdifferential.

1 Some aspects of Rothe's method

Let V be a real Hilbert space, V^* dual space of V, and K a closed convex set in V. Consider the variational inequality

$$\left(\frac{du}{dt}(t), v-u(t)\right) \geq (f, v-u(t)), \qquad \text{for all } v \in K, \tag{1.1}$$

where $u_0 \in K$ and $f \in H^1(0,T;V^*)$, $T>0$, such that $u(t) \in K$ a. e. $t \in (0,T)$.

To analyze the solvability of this problem, we use discrete schemes reducing the above problem to solve some elliptic variational inequalities.

Let n be a positive integer and let $\{t_i\}$ be a partition of $I=(0,T)$ with the step $h=\dfrac{T}{n}$, i.e. $t_i = ih$ for $i=0,1,.,n$. We have associated with (1) the following approximation scheme:

$$\left(\frac{u_i-u_{i-1}}{h}, v-u_i\right) \geq (f_i, v-u_i), \text{ for all } v \in K \text{ where}$$

$f_i = \displaystyle\int_{t_{i-1}}^{t_i} f(s)ds$, $u_i \in K$ ($u_i \in K$ is known) and construct a sequence $\{u_n\}$ converging to a solution of our problem (1). To this end we use the Rothe's functions

$$u_n(t) = u_{i-1} + (t-t_{i-1})h^{-1}(u_i - u_{i-1}), \text{ for } t_{i-1} \leq t \leq t_i \tag{1.2}$$

and the corresponding step functions

$$\bar{u}_n(t) = u_i, \text{ for } t_{i-1} < t \leq t_i, \ i=1,2,...,n \quad \text{and} \quad \bar{u}_n(0)=u_0. \tag{1.3}$$

2 Formulation and basic results

Let V, H be real Hilbert spaces with norms $\|\cdot\|$, $|\cdot|$, respectively, and with the dense and continuous imbedding $V \hookrightarrow H$. The duality between $v \in V$ and $f \in V^*$, denoted by (f, v), coincides with the scalar product in H for $f \in H$. By " $\longrightarrow$ " and " $\rightharpoonup$ " we denote strong and weak convergence, respectively.

Let $\varphi : V \to \bar{\mathbf{R}}$ be a proper, convex, lower semicontinuous (l.s.c) function and $f \in H^1(0, T; V^*)$.

We shall be concerned with the following problem: find a solution $u \in L_\infty(0, T; V) \cap C(0, T; H)$ of the inclusion

$$\frac{du}{dt}(t) + \partial\varphi(u(t)) \ni f(t), \ u(0) = u_0, \ u_0 \in V. \tag{2.1}$$

Regarding the sequences 1.2 and 1.3 we use some preliminary results.

Lemma 2.1. *(see [2]) Let $(V, \|\cdot\|)$, $(W, \|\cdot\|)$ be reflexive Banach spaces with $V \hookrightarrow W$. If $u_n \to u$ in $C(0, T; W)$ and the estimates $\int\limits_0^T \left\|\frac{du_n}{ds}(s)\right\|_w^2 ds \le C$, $\|\bar{u}_n(t)\| \le C$, for all $t \in (0, T)$ holds for all $n \ge n_0$, then there exists $u \in L_\infty(0, T; V)$ with $\frac{du}{dt} \in L_2(0, T; W)$ (u is differentiable a. e. $t \in (0, T)$) and $u_n(t) \rightharpoonup u(t)$, $\bar{u}_n(t) \rightharpoonup u(t)$ in V for all $t \in (0, T)$, $\frac{du_n}{dt} \rightharpoonup \frac{du}{dt}$ in $L_2(0, T; W)$.*

Moreover, if $\left\|\frac{du_n}{ds}(s)\right\|_w \le C$ a.e. $s \in (0, T)$, then $\|u(t) - u(t')\|_w \le C|t - t'|$ for all $t, t' \in (0, T)$ and $\frac{du}{dt} \in L_\infty(0, T; W)$.

About discretization of the right side we have,

Lemma 2.2. *(see [2]) If $f \in H^q(0, T; H)$ $(q \ge 0)$, then $\sum\limits_{i=1}^n \|\delta_h f_i\|_H^2 h \le C \int\limits_0^t \left\|\frac{df}{dt}(s)\right\|_H^2 ds$ for all $p = 0, ..., q$, where $\delta_h f_i = f_i - f_{i-1}$.*

In the proof of convergence of $\{u_n\}$we make use of

Lemma 2.3. *(see [1]) Let $u \in W^{1,2}(0, T; H)$ such that $u(t) \in D(\partial\varphi)$ a.e. $t \in (0, T)$ and there is $g \in L^2(0, T; H)$ so that $g(t) \in \partial\varphi(u(t))$ a.e. $t \in [0, T]$. Then the function $t \to (u(t))$ is absolute continuous on $[0, T]$ and the formula*

$$\frac{d}{dt}\varphi(u(t)) = \left(h(t), \frac{du}{dt}(t)\right) \ \text{a.e. } t \in (0, T)$$

holds for all $h \in L^2(0, T; H)$ such that $h(t) \in \partial\varphi(u(t))$ a.e. $t \in [0, T]$.

3 Existence result

We are in position to establish the

Theorem 3.1. *Let* $u_0 \in V$ *with* $\varphi(u_0) < \infty$, $f \in H^1(0,T;V^*)$ *and assume there is* $z_0 \in H$ *so that*

$$(z_0, v - u_0) + \varphi(v) - \varphi(u_0) \geq (f(0), v - u_0).$$

Then there exist an unique $u \in L_\infty(0,T;V) \cap C(0,T;H)$ *such that*

$$\frac{du}{dt}(t) + \partial\varphi(u(t)) \ni f(t), \text{ a.e. } t \in (0,T), \text{ where } u(0) = u_0. \tag{P}$$

Proof of the existence. Let n be a natural number, $h = \dfrac{T}{n}$, $t_i = ih$ and $f_i = \displaystyle\int\limits_{t_{i-1}}^{t_i} f(s)ds$. Starting with an initial value $u_0 \in V$ we succesively determine $u_i \in Dom(\varphi)$ for $i = 1, 2, ..., n$ a solution of inclusion

$$\frac{u_i - u_{i-1}}{h} + \partial\varphi(u_i) \ni f_i \tag{3.1}$$

which exists due to Minty's result (see [3]) regarding the maximal monotonicity of subdifferential. Taking into account of the subgradient we obtain

$$\left(\frac{u_i - u_{i-1}}{h}, v - u_i\right) + \varphi(v) - \varphi(u_i) \geq (f_i, v - u_i), \text{ for all } v \in V \tag{3.2}$$

or

$$\left(\frac{u_i}{h}, v - u_i\right) + \varphi(v) - \varphi(u_i) \geq (f_i + \frac{u_{i-1}}{h}, v - u_i), \text{ for all } v \in V. \tag{3.3}$$

Put first $v = u_{i-1}$ in 3.3. In relation 3.3 for rang $i-1$ take $v = u_i$. Adding these inequalities we infer

$$\left(\frac{u_i - u_{i-1}}{h}, u_{i-1} - u_i\right) \geq (f_i - f_{i-1}, u_{i-1} - u_i) + \left(\frac{u_{i-1} - u_{i-2}}{h}, u_{i-1} - u_i\right) \tag{3.4}$$

equivalently with

$$\left(\frac{u_i - u_{i-1}}{h}, u_{i-1} - u_i\right) \leq \left(\frac{u_{i-1} - u_{i-2}}{h}, u_{i-1} - u_i\right) + (f_i - f_{i-1}, u_i - u_{i-1}) \tag{3.5}$$

Multiply 3.5 with $1/h$ and deduce

$$\left(\frac{u_i - u_{i-1}}{h}, \frac{u_i - u_{i-1}}{h}\right) \geq \left(\frac{u_{i-1} - u_{i-2}}{h}, \frac{u_i - u_{i-1}}{h}\right) + (f_i - f_{i-1}, \frac{u_i - u_{i-1}}{h}) \tag{3.6}$$

Denote $\delta_h u_i = \dfrac{u_i - u_{i-1}}{h}$ and deduce

$$\begin{aligned} |\delta_h u_i|^2 &\leq (\delta_h u_{i-1}, \delta_h u_i) + (f_i - f_{i-1}, \delta_h u_i) \\ &\leq \frac{1}{2}|\delta_h u_{i-1}|^2 + \frac{1}{2}|\delta_h u_i|^2 + \frac{1}{2h}\|f_i - f_{i-1}\|^2 + \frac{h}{2}|\delta_h u_i|^2 . \end{aligned} \tag{3.7}$$

and

$$(1-h)\left|\delta_h u_i\right|^2 \leq \left|\delta_h u_{i-1}\right|^2 + \frac{1}{2h}\left\|f_i - f_{i-1}\right\|^2 \tag{3.8}$$

From this recurrent inequality we estimate

$$(1-h)^{i-1}\left|\delta_h u_i\right|^2 \leq \left|\delta_h u_1\right|^2 + \frac{1}{h}\sum_{j=1}^{i-1}\left\|f_{j+1} - f_j\right\|^2 \tag{3.9}$$

equivalently with

$$(1-h)^{i-1}\left|\delta_h u_i\right|^2 \leq \left|\delta_h u_1\right|^2 + \frac{1}{h}\sum_{j=2}^{i}\left\|\delta_h f_j\right\|^2 \tag{3.10}$$

As $(1-h)^{i-1} = \left(1 - \frac{T}{n}\right)^{i-1} = \left(\left(1 - \frac{T}{n}\right)^n\right)^{\frac{i-1}{n}} \geq \left(1 - \frac{T}{n}\right)^n \rightarrow e^{-T}$, as $n \rightarrow \infty$, we conclude

$$\left|\delta_h u_i\right|^2 \leq C\left(\left|\delta_h u_1\right|^2 + h\sum_{j=2}^{i}\left\|\delta_h f_j\right\|^2\right). \tag{3.11}$$

Now we prove that $|\delta_h u_i|$ is bounded . Inequality 3.3 for $i = 1$ and $v = u_0$ gives:

$$\frac{1}{2}\left|\delta_h u_i\right|^2 \leq \frac{1}{2}\left\|f_i\right\|^2 + (\varphi(u_1) - \varphi(u_0))h^{-1}. \tag{3.12}$$

Combine 3.11 and 3.12, it follows that

$$\left|\delta_h u_i\right|^2 \leq C\left(\left\|f_1\right\|^2 + C_1 + h\sum_{j=2}^{i}\left\|\delta_h f_j\right\|^2\right) \tag{3.13}$$

which leads to

$$\left|\delta_h u_i\right| \leq C \quad \text{for } i = 1, 2, ..., n. \tag{3.14}$$

On the other hand the inequality 3.6 yields

$$\left|\delta_h u_i\right|^2 \leq \left|\delta_h u_i\right| \cdot \left|\delta_h u_{i-1}\right| + \left\|f_i - f_{i-1}\right\| \cdot \left|\delta_h u_i\right| + \left\|f_i - f_{i-1}\right\| \cdot \left|\delta_h u_i\right| \tag{3.15}$$

or

$$\left|\delta_h u_i\right| \leq \left|\delta_h u_{i-1}\right| + \left\|f_i - f_{i-1}\right\|$$

that means

$$\left|u_i - u_{i-1}\right| \leq \left|u_{i-1} - u_{i-2}\right| + h\left\|f_i - f_{i-1}\right\|$$

Since $|u_i - u_{i-1}| \geq |u_i| - |u_{i-1}|$ we get that

$$\left|u_i\right| \leq 2\left|u_{i-1}\right| + \left|u_{i-2}\right| + h\left\|f_i - f_{i-1}\right\| \tag{3.16}$$

and therefore

$$|u_i| \leq |u_{i-1}| + 2\sum_{j=1}^{i-2} |u_j| + 3\,|u_1| + |u_0| + h\sum_{j=1}^{i} \|f_j - f_{j-1}\| \tag{3.17}$$

Apply Gronwall's Lemma (see [6]) and Lemma 2.2 we conclude the estimates:

$$|u_i| \leq C \quad \text{for } i = 1, 2, ..., n \tag{3.18}$$

and then

$$\|ui\| \leq C \quad \text{for } i = 1, 2, ..., n \tag{3.19}$$

as a consequence of 3.3 and 3.4 .

By means of the above estimates we construct Rothe's functions and corresponding step functions

$$u_n(t) = u_{i-1} + (t - t_{i-1})h^{-1}(u_i - u_{i-1}) \quad \text{for } t_{i-1} \leq t \leq t_i, \quad i = 1, ..., n \tag{3.20}$$

$$\bar{u}_n(t) = u_i \quad \text{for } t_{i-1} < t \leq t_i, \quad i = 1, ..., n \text{ and } \bar{u}_n(0) = u_0. \tag{3.21}$$

Estimates 3.14 and 3.19 for these functions can be rewrite

$$\left|\frac{du_n}{dt}(t)\right| \leq C \quad \text{a.e. } t \in (0, T), \quad \|\bar{u}_n(t)\| \leq C \text{ for } n \geq n_0,\ t \in (0, T) \tag{3.22}$$

and the scheme 3.3 becomes

$$\left(\frac{du_n}{dt}(t), v - \bar{u}_n(t)\right) + \varphi(v) - \varphi(\bar{u}_n(t)) \geq \left(\bar{f}_n(t), v - \bar{u}_n(t)\right), \quad \text{for all } v \in V, \tag{3.23}$$

a.e. $t \in (0, T)$ where $\bar{f}_n(t) = f_i$ for $t_{i-1} < t \leq t_i$.

Put first $v = \bar{u}_m(t)$ in 3.23. In relation 3.23 for rang m take $v = \bar{u}_n(t)$. Adding these inequalities we obtain

$$\left(\frac{du_n}{dt}(t) - \frac{du_m}{dt}(t), \bar{u}_n(t) - \bar{u}_m(t)\right) \leq \left(\bar{f}_n(t) - \bar{f}_m(t), \bar{u}_n(t) - \bar{u}_m(t)\right). \tag{3.24}$$

It follows by computation that

$$\frac{1}{2}\frac{d}{dt}\,|u_n - u_m|^2 \leq \left(\bar{f}_n(t) - \bar{f}_m(t), \bar{u}_n(t) - \bar{u}_m(t)\right) \tag{3.25}$$

and

$$|u_n(t) - u_m(t)| \leq 2\int_0^t \left(\bar{f}_n(s) - \bar{f}_m(s), \bar{u}_n(s) - \bar{u}_m(s)\right) ds \tag{3.26}$$

$$\leq \frac{1}{\varepsilon}[\int_0^t \left\|\bar{f}_n(s) - f(s)\right\|^2 ds + \int_0^t \left\|\bar{f}_m(s) - f(s)\right\|^2 ds] + \varepsilon \int_0^t \|\bar{u}_n(s) - \bar{u}_m(s)\|\, ds.$$

By Lemma 2.2 after simple calculus we deduce

$$|u_n(t)-u_m(t)| \le \frac{1}{\varepsilon}[\int_0^t \left\|\bar{f}_n(s)-f(s)\right\|^2 ds + \int_0^t \left\|\bar{f}_m(s)-f(s)\right\|^2 ds] + \tag{3.27}$$

$$+\varepsilon[\int_0^t |u_n(s)-\bar{u}_n(s)|^2 ds + \int_0^t |u_m(s)-\bar{u}_m(s)|^2 ds + \int_0^t |u_n(s)-u_m(s)|^2 ds].$$

Taking into account the form of Rothe's function, we have

$$|u_n(t)-\bar{u}_n(t)| \le \frac{T}{n}|\delta_h u_i| \le \frac{C}{n} \quad \text{for } t_{i-1} < t \le t_i,\ i=1,...,n \tag{3.28}$$

By 3.24, Gronwall's Lemma and Lemma 2.2 we obtain

$$u_n \to u \text{ in } C(0,T;H), \quad \bar{u}_n \to u \text{ in } L_2(0,T;V) \tag{3.29}$$

Thereby, estimates 3.14, 3.19 and Lemma 2.1 imply

$$\begin{aligned} u \in\ & L_\infty(0,T;V), \frac{du}{dt} \in L_\infty(0,T;H), \\ \bar{u}_n(t) &\rightharpoonup u(t) \text{ in } V \text{ for all } t \in (0,T), \\ \frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L_2(0,T;H). \end{aligned} \tag{3.30}$$

Now using Lemma 2.3 the inequality 3.2 can be rewrite

$$\int_{t_1}^{t_2} [\frac{du_n}{dt}(t), v-\bar{u}_n(t)]dt + \int_{t_1}^{t_2} (\varphi(v)-\varphi(\bar{u}_n(t)))\,dt \ge \int_{t_1}^{t_2} [\bar{f}_n(t), v-\bar{u}_n(t)]dt. \tag{3.31}$$

By virtue of the above estimates we deduce:

$$\bar{u}_n(t) \rightharpoonup u(t) \text{ in } V \text{ for all } t \in (0,T) \text{ and } |u_n(t)-\bar{u}_n(t)| \le \frac{T}{n} \sup_{1\le i\le n} |\delta_h u_i| < \frac{C}{n}$$

and we derive

$$\begin{aligned} \int_{t_1}^{t_2} [\frac{du_n}{dt}(t), v-\bar{u}_n(t)]dt &\to \int_{t_1}^{t_2} [\frac{du}{dt}(t), v-u(t)]dt \\ \int_{t_1}^{t_2} [\bar{f}_n(t), v-\bar{u}_n(t)]dt &\to \int_{t_1}^{t_2} [f(t), v-u(t)]dt. \end{aligned} \tag{3.32}$$

By 3.2 we have $|\varphi(\bar{u}_n(t))| \le C$ and moreover $|\varphi(\bar{u}_n(t))| \le C$ for all $t \in (0,T)$.

This estimate and Fatou's Lemma yields

$$\int_{t_1}^{t_2} \varphi(u(t))dt \leq \liminf_{n\to\infty} \int_{t_1}^{t_2} \varphi(\bar{u}(t))dt \leq C < \infty.$$

Returning to 3.29, because t_1, $t_2 \in (0,T)$ are arbitrarily we conclude that $u(t)$ is a solution of the problem (P).

Proof of the uniqueness. For two solutions u_1 and u_2 corresponding to f_1 and f_2 we have :

$$(f_2 - f_1, u_2 - u_1) \geq \left(\frac{du_2}{dt} - \frac{du_1}{dt}, u_2 - u_1\right) \tag{3.33}$$

and

$$\frac{1}{2}\int_0^t \frac{d}{ds}\left|u_2 - u_1\right|^2 ds \leq \int_0^t (f_2 - f_1, u_2 - u_1)ds \tag{3.34}$$

whence the uniqueness follows at once. □

References

1. Brezis H., *Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert*, North - Holland / American Elsevier, Amsterdam, Holland, 1973.
2. Kaèur J., *Method of Rothe in Evolution Equations*, Teubner- Texte zur Mathematik- Bd. 80, Leipzig, 1985.
3. Minty G.J., *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. 29 (1962), pp. 341-346.
4. Rektorys K., *The Method of Discretization in Time and Partial Differential Equations*, D. Reidel Publishing Company, Dordrecht, Holland, 1982.
5. Rothe E., *Zweidimensionale parabolische Randwertaufgahen als Grenfall eindimensionaler Randwertaufgaben*, Math. Ann, 102 (1930).
6. Quarteroni A., *Numerical Approximation of Partial Differential Equations*, Springer - Verlag, Berlin Heidelberg, 1994.