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NONANALYTIC FUNCTIONS WITH POSITIVE REAL PART AND STARLIKE FUNCTIONS

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Abstract. It is well-known that if p is an analytic function in the unit disc U, with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, for $z \in U$, then the function $f(z) = z + \cdots$ satisfying the equation $zf'(z)/f(z) = p(z)$ is starlike in U. In this paper we give an extention of this result in the nonanalytic case.

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1. Let U, with $U^* = U\backslash\{0\}$, be the unit disc of the complex plane $\mathbb{C}$ and let denote by $C^1(U)$ the class of complex functions f for which the real functions $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuously differentiable on U.

For $f \in C^1(U)$ and $z = x + iy \in U$ let define the differential operators D and $\mathcal{D}$ by

$$Df(z) = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}} \quad \text{and} \quad \mathcal{D}f(z) = z\frac{\partial f}{\partial z} + \overline{z}\frac{\partial f}{\partial \overline{z}}, \tag{1}$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right)$$

The Jacobian of f is given by

$$Jf(z) = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2 = \frac{1}{|z|^2}\operatorname{Re}\left[Df(z)\overline{\mathcal{D}f(z)}\right]. \tag{2}$$

The function $f \in C^1(U)$, with $f(0) = 0$ is said to be starlike if f is injective and $f(U)$ is starlike with respect to the origin.

If f is analytic in U, then it is well known that f is starlike if and only if $f'(0) \neq 0$ and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0. \tag{3}$$

The following theorem provides a sufficient condition for starlikeness for functions in $C^1(U)$.

Theorem 1. [1] *If $f \in C^1(U)$ satisfies the conditions:*

(i) $f(0) = 0$, $f(z) \neq 0$ *for* $z \in U^*$,
(ii) $Jf(z) > 0$, *for* $z \in U$,

(*iii*) $\operatorname{Re}\frac{Df(z)}{f(z)} > 0$, *for* $z \in U^*$,

then f *is starlike.*

For $f \in C^1(U)$, with $f(0) = 0$, we can write

$$f(z) = Az + B\overline{z} + o(z), \quad z \in U$$

and

$$Df(z) = Az - B\overline{z} + o(z), \quad z \in U,$$

where Df is given by (1) and

$$A = \frac{\partial f}{\partial z}(0), \quad B = \frac{\partial f}{\partial \overline{z}}(0).$$

If $f(z) \neq 0$ for $z \in U^*$, then we can define the function $p \in C^1(U^*)$ by

$$p(z) = \frac{Df(z)}{f(z)}, \quad z \in U^*$$

and we have

$$p(z) = \frac{Az - B\overline{z}}{Az + B\overline{z}} + O(z), \quad z \in U^*.$$

It is well known that if p is an analytic function in U, with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, for $z \in U$, then the function f given by

$$f(z) = z \exp \int_0^z \frac{p(\zeta) - 1}{\zeta} d\zeta$$

is starlike, i.e. it satisfies (3).

2. An extension of the above result in the nonanalytic case is given by the following theorem

Theorem 2. *Let* $p \in C^1(U^*)$ *and suppose that for all* $z \in U^*$ *the following equality holds*

$$p(z) = \frac{Az - B\overline{z}}{Az + B\overline{z}} + q(z), \tag{4}$$

where $A, B \in \mathbb{C}$, *with* $|A| > |B|$ *and* $q \in C^1(U)$, *with* $q(0) = 0$, $\int_0^{2\pi} q(re^{it})dt = 0$, *for* $0 < r < 1$.

Let the function $\Phi \in [0,1) \longrightarrow \mathbb{C}$ *satisfy the conditions* $\Phi \in C^1[0,1)$, $\Phi(0) = 0$, $\Phi'(0) = 1, \Phi(r) \neq 0$, *for* $0 < r < 1$, Φ' *is differentiable at* 0 *and*

$$\Phi''(0) = 2\left[\frac{\partial q}{\partial z}(0) - \frac{\partial q}{\partial \overline{z}}(0)\right].$$

Then

(i) the function $f : U \to \mathbb{C}$ defined by

$$f(z) = \frac{\Phi(r)}{r}\frac{Az + B\overline{z}}{A+B}\exp\int_0^\theta iq(re^{it})dt, \quad z = re^{i\theta} \in U^*, \tag{5}$$

with $f(0) = 0$, belongs to $C^1(U)$ and satisfies the equation

$$\frac{Df(z)}{f(z)} = p(z), \quad z \in U^*, \tag{6}$$

with the conditions

$$f(r) = \Phi(r), \quad 0 < r < 1, \quad \frac{\partial f}{\partial z}(0) = \frac{A}{A+B}, \quad \frac{\partial f}{\partial \overline{z}}(0) = \frac{B}{A-B}$$

(ii) moreover, if the following conditions hold

$$\operatorname{Re} p(z) > 0, \quad z \in U^* \tag{7}$$

and

$$\operatorname{Re}\left\{\overline{p(re^{i\theta})}\left[\frac{r\Phi'(r)}{\Phi(r)} + ir\int_0^\theta \frac{\partial}{\partial r}p(re^{it})dt\right]\right\} > 0, \quad z = re^{i\theta} \in U^*, \tag{8}$$

then f is starlike.

PROOF. The function $\varphi : [0,1) \to \mathbb{C}$ be given by

$$\varphi(r) = \log\frac{\Phi(r)}{r}, \quad r \in (0,1), \ \varphi(0) = 0, \tag{9}$$

where the function log is defined by continuity on the path $w = \Phi(r)/r$, $r \in (0,1)$, $w(0) = 1$ and $\log 1 = 0$. It is obvious that $\varphi \in C[0,1) \cap C^1(0,1)$. We also have

$$\Phi(r) = r + \frac{\Phi''(0)}{2}r^2 + o(r^2), \quad r \in (0,1)$$

$$\Phi'(r) = 1 + \Phi''(0)r + o(r), \quad r \in (0,1)$$

which yield

$$\frac{\Phi'(r)}{\Phi(r)} = \frac{1}{r} + \frac{\Phi''(0)}{2} + O(r), \quad r \in (0,1).$$

Since for $r \in (0,1)$ we have

$$\varphi'(r) = \frac{\Phi'(r)}{\Phi(r)} - \frac{1}{r} = \frac{\Phi''(0)}{2} + O(r)$$

we deduce

$$\lim_{r\to 0}\varphi'(r) = \frac{\Phi''(0)}{2}.$$

On the other hand we have

$$\varphi'(0) = \lim_{r\to 0} \frac{\varphi(r)}{r} = \lim_{r\to 0}\left[\frac{\Phi'(r)}{\Phi(r)} - \frac{1}{r}\right] = \frac{\Phi''(0)}{2}.$$

Therefore $\varphi \in C^1[0,1)$ and

$$\varphi'(0) = \frac{\partial q}{\partial z}(0) - \frac{\partial q}{\partial \overline{z}}(0).$$

Let the function $Q : U \to \mathbb{C}$ be given by

$$Q(z) = \varphi(r) + \mathrm{i}\int_0^\theta q(re^{\mathrm{i}t})dt, \quad z = re^{\mathrm{i}\theta} \in U^*, \quad Q(0) = 0. \tag{10}$$

According to Theorem 3, $Q \in C^1(U)$ and $DQ = q$, $Q(r) = \varphi(r)$. From (4) and (10) we deduce that the function f given by (5) can be written as

$$f(z) = \frac{Az + B\overline{z}}{A+B}e^{Q(z)}, \quad z \in U \tag{11}$$

and we deduce that $f \in C^1(U)$, $f(r) = \Phi(r)$ and

$$\frac{Df(z)}{f(z)} = \frac{Az - B\overline{z}}{Az + B\overline{z}} + DQ(z) = \frac{Az - B\overline{z}}{Az + B\overline{z}} + q(z) = p(z), \quad z \in U^*.$$

Since

$$\frac{\partial f}{\partial z}(0) = \frac{A}{A+B}e^{Q(z)} + \frac{Az + B\overline{z}}{A+B}e^{Q(z)}\frac{\partial Q}{\partial z}$$

we deduce

$$\frac{\partial f}{\partial z}(0) = \frac{A}{A+B}.$$

We also have

$$\frac{\partial f}{\partial \overline{z}}(0) = \frac{B}{A+B}.$$

(ii) Condition (7) becomes

$$\mathrm{Re}\,\frac{Df(z)}{f(z)} > 0, \quad z \in U^*.$$

On the other hand for $z \in U^*$ from (11) we deduce

$$\frac{\mathcal{D}f(z)}{f(z)} = 1 + \mathcal{D}Q(z)$$

and by using (8) and (10) we obtain

$$\mathrm{Re}\left[\left(\overline{\frac{Df(z)}{f(z)}}\right)\frac{\mathcal{D}f(z)}{f(z)}\right] = \mathrm{Re}[\overline{p(z)}(1 + \mathcal{D}Q(z))]$$

$$= \mathrm{Re}\left\{\overline{p(re^{\mathrm{i}\theta})}\left[1 + r\varphi'(r) + \mathrm{i}r\int_o^\theta \frac{\partial}{\partial r}q(re^{\mathrm{i}t})dt\right]\right\}$$

$$= \mathrm{Re}\left\{\overline{p(re^{\mathrm{i}\theta})}\left[\frac{r\Phi'(r)}{\Phi(r)} + \mathrm{i}r\int_o^\theta \frac{\partial}{\partial r}q(re^{\mathrm{i}t})dt\right]\right\} > 0.$$

According to (2) this last inequality is equivalent to $Jf(z) > 0$, for $z \in U^*$.We also have

$$Jf(0) = \frac{|A|^2 - |B|^2}{|A+B|^2} > 0.$$

Hence $Jf(z) > 0$ for all $z \in U$ and f satisfies all conditions of Theorem 1. Therefore we deduce that f is starlike.

3. In the case $B = 0$ Theorem 2 can be improved as follows.

Theorem 3. *Let $p \in C^1(U^*)$ and suppose that*

$$\begin{cases} \lim\limits_{z\to 0} p(z) = 1 \\ \lim\limits_{r\to 0} r\displaystyle\int_o^\theta \frac{\partial}{\partial r}p(re^{it})dt = o, \text{ uniformly on } [0,1], \\ \displaystyle\int_0^{2\pi}[p(re^{it}) - 1]dt = 0, \quad 0 < r < 1 \end{cases} \tag{12}$$

Let the function $\Phi : [0,1) \to \mathbb{C}$ satisfy the conditions $\Phi \in C^1[0,1)$, $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi(r) \neq 0$, for $0 < r < 1$.

Then

(*i*) *the function $f : U \to \mathbb{C}$ given by*

$$\begin{aligned} f(z) &= \frac{\Phi(r)}{r} z \exp\int_0^\theta \mathrm{i}[p(re^{it}) - 1]dt \\ &= \Phi(r)\exp\int_0^\theta \mathrm{i}p(re^{\mathrm{i}t})dt, \quad z = re^{\mathrm{i}\theta} \in U^* \end{aligned} \tag{13}$$

and $f(0) = 0$, belongs to $C^1(U)$ and satisfies the equation

$$\frac{Df(z)}{f(z)} = p(z), \quad z \in U^*,$$

with the conditions $f(r) = \Phi(r)$, $0 \le r < 1$ and

$$\frac{\partial f}{\partial z}(0) = 1, \quad \frac{\partial f}{\partial \bar{z}}(0) = 0.$$

(*ii*) *moreover, if conditions (7) and (8) hold then f is starlike.*

PROOF. The function φ given by (9) and let

$$Q(z) = \varphi(r) + \mathrm{i}\int_0^\theta [p(re^{\mathrm{i}t}) - 1]dt, \quad z = re^{\mathrm{i}\theta} \in U^*, \quad Q(0) = 0.$$

It is obvious that $Q \in C^1(U^*)$ and since $\lim_{z\to 0} Q(z) = 0$, the function Q is continuous on U.

The function f given by (13) can be rewritten as

$$f(z) = z \exp Q(z), \quad z \in U.$$

Hence we have $f \in C^1(U^*)$. Also, since $\lim_{z\to 0} \frac{f(z)}{z} = 1$, we deduce that f is differentiable at 0 and $\frac{\partial f}{\partial z}(0) = 1$, $\frac{\partial f}{\partial \overline{z}}(0) = 0$.

For $z = re^{i\theta} \in U^*$ we have

$$\begin{aligned} \frac{\partial f}{\partial z} &= e^{Q(z)} + ze^{Q(z)}\left[\frac{r\varphi'(r)}{2z} + \frac{p(z)-1}{2z} + \frac{\mathrm{i}r}{2z}\int_0^\theta \frac{\partial}{\partial r} p(re^{\mathrm{i}t})dt\right] \\ &= e^{Q(z)} + \frac{1}{2}e^{Q(z)}\left[r\varphi'(r) + p(z) - 1 + \mathrm{i}r\int_0^\theta \frac{\partial}{\partial r} p(re^{\mathrm{i}t})dt\right]. \end{aligned}$$

Using conditions (12) and the fact that

$$\lim_{r\to 0} r\varphi'(r) = \lim_{r\to 0} \frac{r\Phi'(r)}{\Phi(r)} - 1 = 0,$$

we deduce

$$\lim_{z\to 0} \frac{\partial f}{\partial z}(z) = \frac{\partial f}{\partial z}(0) = 1.$$

We also have

$$\lim_{z\to 0} \frac{\partial f}{\partial \overline{z}}(z) = \frac{\partial f}{\partial \overline{z}}(0) = 0.$$

Hence $f \in C^1(U)$ and

$$\frac{Df(z)}{f(z)} = p(z), \text{ for } z \in U^*,$$

$$f(r) = \varphi(r), \quad \frac{\partial f}{\partial z}(0) = 1, \quad \frac{\partial f}{\partial \overline{z}}(0) = 0.$$

(ii) The proof is similar to the case of Theorem 2.

References

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