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SOME SUBCLASSES OF REGULAR UNIVALENT FUNCTIONS

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Abstract. The aim of this investigation is to obtain a new univalence criterion for the regular functions defined in the unit disc U, which is based on the well known Becker's univalence criterion, but which does not depend on $|z|$.
Using this criterion we can define a set of new subclasses of univalent functions which were called pseudo-convex of order θ.
Moreover, we give also univalence criteria in connection with the integral operators introduced by H.Ovesea, N.N.Pascu, V.Pescar, through improvement of similar results obtained by Y.J.Kim, E.P.Merkes and J.P.Pfaltzgraff.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f analytic in the unit disk U normalized by $f(0) = 0$ and $f'(0) = 1$. We note with S the subclass of $\mathcal{A}$ containing the univalent functions.

It is useful to mention some theorems with reference to the properties of the integral operators which preserve the univalence.

Theorem 1.1. *[5] Let $f \in S$, $n \in N$, $\alpha \in C$, $|\alpha| < \frac{1}{4}$ then the function defined by*

$$F_{\alpha,n}(z) = \int_0^z [f'(u^n)]^\alpha du \qquad (\forall)\ z \in U$$

belongs also to the class S.

Theorem 1.2. *[4]. Let $f \in S$, $n \in N$, $\beta \in C$. If $|\beta| \leq \frac{1}{4n}$ then the function*

$$F_{\beta,u}(z) = \int_0^z \left[\frac{f(u^n)}{u^n}\right]^\beta du \qquad z \in U$$

is also univalent.

Theorem 1.3. *Let $f \in S$, $n \in N$, $\beta, \alpha \in C$.*

If $|\beta| + |\alpha| \leq \frac{1}{4n}$ then the function, defined as

$$F_{\alpha,\beta,u}(z) = \int_0^z \left[\frac{f(u^n)}{u^n}\right]^\beta [f'(u^n)]^\alpha du \qquad z \in U$$

is univalent.

Remark. For $n = 0$, $\beta = 0$, $\alpha \neq 0$ Theorem 1.3 is reduced to a theorem given by Pfaltzgraff Y. in 1975 [6] and for $n = 0$, $\beta \neq 0$, $\alpha = 0$ one obtains another result due by Kim Y.J. and Merkes E.P. [3].

Now we consider the Caratheodory inequality which is based on the Schwarz's Lemma.

Lemma 1.1 (Caratheodory). *Let $g(z) \in \mathcal{H}(U)$, $g(0) = 0$, M real pozitiv number. If for $z \in U$*

$$Re\ g(z) \leq M$$

then

$$(1 - |z|)|g(z)| \leq 2M|z| \qquad \forall\ z \in U.$$

PROOF. Let be the function

$$h(z) = \frac{g(z)}{2M - g(z)}.$$

that is $h(0) \in \mathcal{H}(U)$ and $|h(z)| \leq 1$ because

$$|g(z)| \leq |2M - g(z)|$$

According to the Schwarz's lemma we have:

$$|h(z)| \leq |z| \qquad (\forall)\ z \in U$$

that is

$$|g(z)| \leq |z||2M - g(z)| \leq |z|(2M + |g(z)|).$$

Finally

$$(1 - |z|)|g(z)| \leq 2M|z| \qquad \forall\ z \in U$$

which complete the proof. □

2 Main results

In the following we give an univalence criterion which is based on the Becker's criterion but it is more practical.

Theorem 2.1. *[2] Let $f \in A$. If*

$$Re\left[e^{i\theta}\frac{zf''(z)}{f'(z)}\right] \leq \frac{1}{4}$$

then $f \in S$ $(\forall)$ $\theta \in [0, 2\pi]$.

PROOF. In lemma 1.1 we take

$$g(z) = e^{i\theta}\frac{zf''(z)}{f'(z)}$$

and we have

$$(1 - |z|)\left|\frac{zf''(z)}{f'(z)}\right| \leq 2\frac{1}{4}|z| = \frac{|z|}{2}$$

on the other hand

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| = (1+|z|)(1-|z|)\left|\frac{zf''(z)}{f'(z)}\right| \leq (1+|z|)\frac{|z|}{2} \leq 1$$

According to the Becker's univalence criterion [1] it follows that $f \in S$. □

SPECIAL CASES. The above theorem contains in fact a set of univalence criteria depending on parameter $\theta \in [0, 2\pi]$.

a) if $\theta = 0$ we obtain $Re\frac{zf''(z)}{f'(z)} \leq \frac{1}{4}$ implies $f \in S$;

b) if $\theta = \pi$, $Re\frac{zf''(z)}{f'(z)} \geq -\frac{1}{4}$ also $f \in S$.

The last condition can be written as $Re\frac{zf''(z)}{f'(z)} + 1 \geq \frac{3}{4}$.

The case b is a weaker criterion compared with the criterion for the convex function, but other criteria obtained for other values of the parameter θ are independent and useful.

c) for instance for $\theta = \frac{\pi}{2}$ we obtain

$$Re\ i\frac{zf''(z)}{f'(z)} = -Im\frac{zf''(z)}{f'(z)} \leq \frac{1}{4} \quad ; \quad Im\frac{zf''(z)}{f'(z)} \geq -\frac{1}{4}$$

Mainly, if the differential expression $\frac{zf''(z)}{f'(z)}$, for $(\forall)\ z \in U$, takes values in a half plane which includes the disk $|z| \leq \frac{1}{4}$ so that, it is tangent at the boundary then f is univalent ($f \in S$). We note with $S_\theta^{1/4}$ this half plane.

Definition 2.1. *Let $f \in A$, $(\forall)\ z \in U$. If the differential expression $\frac{zf''(z)}{f'(z)}$ takes all values in the half plane $S_\theta^{1/4}$ then we say that f is a* ***pseudo convex function of θ–type*** *and we note with PC_θ the class of all these functions, for every $\theta \in [0, 2\pi]$.*

Corollary 2.1. *If $f \in PC_\theta$, $(\forall)\ \theta \in [0, 2\pi]$ then f is univalent.*

PROOF. According to the theorem 1 and the above definition the conclusion of corollary 1 follows. □

Remark. $PC_\pi = C(3/4) \subset C$ where we note with C the class of convex function and with $C(3/4)$ the class of convex functions of order $3/4$.

If $f \in PC_\pi$, $Re\frac{zf''(z)}{f'(z)} \geq -\frac{1}{4}$ that is

$$Re\frac{zf''(z)}{f'(z)} + 1 \geq \frac{3}{4} \Rightarrow f \in C(3/4)$$

It is easy to observe that the function

$$K(z,\theta) = \frac{e^{i\theta}}{2} \cdot \frac{z}{1+z} = \frac{e^{i\theta}}{2}\ (z - z^2 + z^2 - \ldots)$$

maps the unit disk U in the half plane $Re\ e^{i\theta}w < \frac{1}{4}$.

Using the subordination concept, we can give an equivalent definition with the definition 2.1.

Definition 2.2. *Let $f \in A$. f is a pseudoconvex function, $f \in PC_\theta$ if and only if*

$$\frac{zf''(z)}{f'(z)} \prec K(z,\theta) = \frac{e^{i\theta}}{2} \cdot \frac{z}{z+1}$$

for every $\theta \in [0, 2\pi]$.

According to the above definition we obtain several coefficient bounds.

Theorem 2.2. *If $f \in PC_\theta$ and*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

then

$$|a_2| \leq \frac{1}{4},\ |a_3| \leq \frac{1}{8},\ |a_4| \leq \frac{5}{64},\ |a_5| \leq \frac{7}{128},\ |a_6| \leq \frac{21}{512}$$

and

$$|a_n| \leq \frac{a}{2n(n-1)}(1 + 2|a_2| + 3|a_3| + \cdots + (n-1)|a_n|).$$

For the proof of this theorem we need the following result by Rogosinski.

Theorem 2.3 (Rogosinski [7]). *Let $h(z) = \sum_{k=1}^{\infty} c_k z^k$ be subordinate to $H(z) = \sum_{k=1}^{\infty} b_k z^k$ in U. If $H(z)$ is univalent in U and $H(U)$ is convex then $|c_n| \leq |b_1|$.*

PROOF. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in PC_\theta$ and we define

$$\frac{zf''(z)}{f'(z)} = \varphi(z) = \sum_{k=1}^{\infty} c_k z^k.$$

Then $\varphi(z) \prec K(z,\theta)$ ($\prec$ denotes subordination).

$K(z,\theta)$ is univalent in U and $K(U,\theta)$, the half plane $Re\ e^{i\theta} n < \frac{1}{4}$, is a convex region, so the Rogosinski's theorem applies.

Now writing $zf''(z) = f'(z) \cdot \varphi(z)$ and comparing coefficients of z^{n-1} on the both sides, we get

$$n(n-1)a_n = \sum_{k=1}^{n-1} kC_{n-k} z_k, \text{ when } a_1 = 1 \tag{2.1}$$

According to the Rogosinski's theorem and the series expansion of the function $K(z,\theta)$ we get $|c_n| \leq \frac{1}{2}$. Using (2.1), with a simple computation one obtain the conclusion of the theorem 2.2. □

Theorem 2.4. *Let be $f \in A$ and $\alpha \in C$, $|\alpha| \leq \frac{1}{4n}$, $n \in \mathbb{N}$.*
If

$$Re\ e^{i\theta}\frac{zf''(z)}{f'(z)} \leq \frac{1}{4}$$

then the function $F_{\alpha,n}(z)$ defined by

$$F_{\alpha,n}(z) = \int_0^z [f'(u^n)]^\alpha du \qquad z \in U \tag{2.2}$$

is univalent for every $\theta \in [0, 2\pi]$

Proof. From Theorem 2.1 follows that the function $f(z)$ is univalent and according to Theorem 1.1 all functions given by (2.2) are also univalent. □

Theorem 2.5. *Let $f \in A$, $n \in \mathbb{N}$, $\beta \in C$, $|\beta| \leq \frac{1}{4n}$.*
If

$$Re\ e^{i\theta}\frac{zf''(z)}{f'(z)} \leq \frac{1}{4}$$

then

$$F_{\beta,n}(z) = \int_0^z \left(\frac{f(u^n)}{u^n}\right)^\beta du \tag{2.3}$$

is univalent in U, $(\forall)\ \theta \in [0, 2\pi]$.

Theorem 2.6. *Let $f \in A$, $n \in \mathbb{N}$, $\alpha, \beta \in C$, $|\alpha| + |\beta| \leq \frac{1}{4n}$.*
If

$$Re\ e^{i\theta}\frac{zf''(z)}{f'(z)} \leq \frac{1}{4}$$

then the functions

$$F_{\alpha,\beta,n}(z) = \int_0^z \left(\frac{f(u^n)}{u^n}\right)^\beta \cdot \left(f'(u^n)\right)^\alpha du \tag{2.4}$$

Proofs for Theorems 2.5 and 2.6 are analogous with that form Theorem 2.4.

Using the conclusion of the Theorem 2.1 and Theorem 1.2 respectively 1.3 it follows the conclusion of Theorems 2.5 and 2.6.

Remark. In the Theorems 2.4, 2.5, 2.6 for the functions as

$$\left[\frac{f(u^n)}{u^n}\right]^\beta \quad \text{or} \quad [f'(u^n)]^\alpha$$

we can choose the regular branch which is equal to 1 at the origin.

In all this univalence criteria the hypothesis included conditions which does't of $|z|$ that is this theorems can be easily applicated.

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